

TABLE 7.2  
How to integrate odd positive powers of sines and cosines

To evaluate	Write	Substitute
$\int \sin^{2n+1} x \, dx$	$(\sin^2 x)^n \sin x = (1 - \cos^2 x)^n \sin x \, dx$	$u = \cos x$ $du = -\sin x \, dx$
$\int \cos^{2n+1} x \, dx$	$(\cos^2 x)^n \cos x = (1 - \sin^2 x)^n \cos x \, dx$	$u = \sin x$ $du = \cos x \, dx$

## PROBLEMS

Evaluate the integrals in Problems 1–22. In evaluating the definite integrals, you may wish to refer to the triangles in Fig. 7.9.

1.  $\int \sin^5 x \, dx$

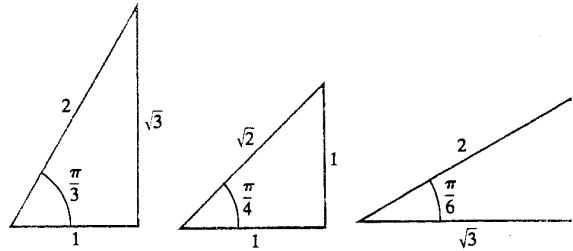
2.  $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$

3.  $\int_{-\pi/2}^{\pi/2} \cos^3 x \, dx$

4.  $\int \cos^5 3x \, dx$

5.  $\int \sin^7 x \, dx$

6.  $\int \cos^7 x \, dx$



7.9 Reference triangles for evaluating trigonometric functions at  $\pi/6$ ,  $\pi/4$ , and  $\pi/3$  radians.

7.  $\int \cos^{2/3} x \sin^5 x \, dx$

(Hint: Put one factor of  $\sin x$  with  $dx$  and express the rest in terms of the cosine.)

8.  $\int \sin^{3/2} x \cos^3 x \, dx$

(Hint: Put one factor of  $\cos x$  with  $dx$  and express the rest in terms of the sine.)

9.  $\int_0^{\pi/3} \sec x \, dx$

10.  $\int \sec 4t \, dt$

11.  $\int_{-\pi/3}^0 \sec^3 x \, dx$

12.  $\int_0^{\pi/6} \frac{2 \sin^2 x}{\cos x} \, dx$

13.  $\int e^x \sec^3 e^x \, dx$

14.  $\int_0^{\pi/4} \sec^4 x \, dx$

15.  $\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^4 x}$

16.  $\int \sec^4 3x \, dx$

17.  $\int_0^{\pi/4} \sec^8 x \, dx$

18.  $\int \frac{2x \, dx}{\cos^3(x^2)}$

19.  $\int_0^{\pi/4} \tan^3 x \, dx$

20.  $\int_{-\pi/4} \tan^4 x \, dx$

21.  $\int \tan^6 x \, dx$

22. a)  $\int \cot^2 x \, dx$

b)  $\int \cot^4 x \, dx$

23. Show that

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C.$$

(Hint: Repeat the short derivation of the integral of the secant with cofunctions.)

24. Use the result in Problem 23 to show that

$$\int \csc^3 x \, dx = -\frac{1}{2} \csc x \cot x - \frac{1}{2} \ln|\csc x + \cot x| + C.$$

25. Evaluate

a)  $\int_0^{\pi/4} \frac{dx}{\sqrt{1 - \sin^2 x}}$

b)  $\int_{\pi/3}^{\pi/2} \frac{dx}{\sqrt{1 - \cos^2 x}}$

26. Evaluate

$$\int_0^{\pi/3} \frac{dx}{1 + \sin x}.$$

(Hint: Multiply numerator and denominator by  $1 - \sin x$ .)

27. Evaluate the integrals in Problems 27–32.

27.  $\int \sin 3x \cos 2x \, dx$

28.  $\int \cos 3x \sin 2x \, dx$

29.  $\int_{-\pi}^{\pi} \sin^2 3x \, dx$

30.  $\int_0^{\pi/2} \sin x \cos x \, dx$

31.  $\int_0^{\pi} \cos 3x \cos 4x \, dx$

32.  $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx$

33. Use the result in Problem 23 and the identity  $2 \sin A \cos A = \sin 2A$  to evaluate

$$\int \frac{\sec 2x \csc 2x}{2} \, dx.$$

34. Which integrals are zero and which are not? (You can do most of these without writing anything down.)

a)  $\int_{-\pi}^{\pi} \sin x \cos^2 x \, dx$

b)  $\int_{-1}^1 \sin 3x \cos 5x \, dx$

c)  $\int_{-L}^L \sqrt[3]{\sin x} \, dx$

d)  $\int_{-a}^a x \sqrt{a^2 - x^2} \, dx$

e)  $\int_{-\pi/4}^{\pi/4} x \sec x \, dx$

f)  $\int_{-\pi/4}^{\pi/4} \tan^3 x \, dx$

g)  $\int_{-\pi/2}^{\pi/2} x \sin x \, dx$

h)  $\int_{-\pi/2}^{\pi/2} x \cos x \, dx$

i)  $\int_{-a}^a \sin mx \cos mx \, dx, m \neq 0$

j)  $\int_{-\pi}^{\pi} \sin^5 x \, dx$

k)  $\int_{-\pi/2}^{\pi/2} \cos^3 x \, dx$

l)  $\int_{-\pi}^{\pi} \cos^5 x \, dx$

m)  $\int_{-\ln 2}^{\ln 2} x(e^x + e^{-x}) \, dx$

n)  $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx$

o)  $\int_{-\pi/2}^{\pi/2} \sin x \sin 2x \, dx$

p)  $\int_{-\pi/4}^{\pi/4} \sec x \tan x \, dx$

q)  $\int_{-a}^a (e^x \sin x + e^{-x} \sin x) \, dx$

r)  $\int_{-1}^1 \frac{\sin x \, dx}{e^x + e^{-x}}$

35. Find the area of the region bounded above by  $y = 2 \cos x$  and below by  $y = \sec x$ ,  $-\pi/4 \leq x \leq \pi/4$ .

36. Find the length of the curve

$$y = \ln(\cos x), \quad 0 \leq x \leq \pi/3.$$

37. Find the length of the curve

$$y = \ln(\sec x), \quad 0 \leq x \leq \pi/4.$$

38. Find the centroid of the region bounded by the  $x$ -axis, the curve  $y = \sec x$ , and the lines  $x = -\pi/4$ ,  $x = \pi/4$ .39. **CALCULATOR OR TABLES** How far apart should the following lines of latitude be on the Mercator map of Example 7?a)  $30^\circ$  and  $45^\circ$  north (about the latitudes of New Orleans, La., and Minneapolis, Minn.);b)  $45^\circ$  and  $60^\circ$  north (about the latitudes of Salem, Ore., and Seward, Al.).

40. Show that

a)  $\int_0^{2\pi} \sin mx \sin nx \, dx = 0,$

$$41. \quad (a) \int_k^{k+2\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_k^{k+2\pi} [\cos(m-n)x - \cos(m+n)x] \, dx$$

$$= \left. \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, m^2 \neq n^2 \right]_k^{k+2\pi} = 0$$

since since  $\sin k = \sin(k+2\pi)$

$$(b) \int_k^{k+2\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_k^{k+2\pi} [\cos(m-n)x + \cos(m+n)x] \, dx$$

$$= \left. \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, m^2 \neq n^2 \right]_k^{k+2\pi} = 0 \text{ as in part (a).}$$

## 7.4 EVEN POWERS OF SINES AND COSINES

$$1. \quad \int_{-\pi}^{\pi} \sin^2 x \, dx = \int_{-\pi}^{\pi} \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) dx = \left. \frac{1}{2}x - \frac{1}{4} \sin 2x \right]_{-\pi}^{\pi} = \pi$$

$$3. \quad \int \sin^2 2t \, dt = \int \left( \frac{1}{2} - \frac{1}{2} \cos 4t \right) dt = \frac{1}{2}t - \frac{1}{8} \sin 4t + C$$

$$5. \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x \cos^2 x \, dx = \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2 \sin x \cos x)^2 \, dx = \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 2x \, dx$$

$$= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \frac{1}{2} - \frac{1}{2} \cos 4x \right) dx = \left. \frac{1}{8}x - \frac{1}{32} \sin 4x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{16}$$

$$7. \quad \int_0^{\frac{\pi}{a}} \sin^4 ax \, dx = \int_0^{\frac{\pi}{a}} (\sin^2 ax)^2 \, dx = \int_0^{\frac{\pi}{a}} \left( \frac{1}{2} - \frac{1}{2} \cos 2ax \right)^2 \, dx$$

$$= \int_0^{\frac{\pi}{a}} \left( \frac{1}{4} - \frac{1}{2} \cos 2ax + \frac{1}{4} \cos^2 2ax \right) dx$$

$$= \int_0^{\frac{\pi}{a}} \left[ \left( \frac{1}{4} - \frac{1}{2} \cos 2ax + \frac{1}{4} \left( \frac{1}{2} + \frac{1}{2} \cos 4ax \right) \right) \right] dx$$

$$= \left. \frac{3}{8}x - \frac{1}{4a} \sin 2ax + \frac{1}{32a} \sin 4ax \right]_0^{\frac{\pi}{a}} = \frac{3\pi}{8a}$$

$$9. \quad \int \frac{\sin^4 x}{\cos^2 x} \, dx = \int \frac{\sin^2 x (1 - \cos^2 x)}{\cos^2 x} \, dx = \int (\tan^2 x - \sin^2 x) \, dx$$

$$= \int \left( \sec^2 x - 1 - \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) \right) dx = \tan x - \frac{3}{2}x + \frac{1}{4} \sin 2x + C$$

$$11. \int_0^{\pi} \sin^4 y \cos^2 y \, dy = \int_0^{\pi} \sin^4 y (1 - \sin^2 y) \, dy = I_1 - I_2, \text{ where}$$

$$\begin{aligned} I_1 &= \int_0^{\pi} \sin^4 y \, dy = \int_0^{\pi} \left( \frac{1}{2} - \frac{1}{2} \cos 2y \right)^2 dy \\ &= \int_0^{\pi} \left( \frac{1}{4} - \frac{1}{2} \cos 2y + \frac{1}{4} \cos^2 2y \right) dy \\ &= \int_0^{\pi} \left( \frac{1}{4} - \frac{1}{2} \cos 2y + \frac{1}{4} \left( \frac{1}{2} + \frac{1}{2} \cos 4y \right) \right) dy \\ &= \frac{3}{8} y - \frac{1}{4} \sin 2y + \frac{1}{32} \sin 4y \Big|_0^{\pi} = \frac{3\pi}{8} \end{aligned}$$

$$\begin{aligned} \text{and } I_2 &= \int_0^{\pi} \sin^6 y \, dy = \int_0^{\pi} \left( \frac{1}{2} - \frac{1}{2} \cos 2y \right)^3 dy \\ &= \frac{1}{8} \int_0^{\pi} (1 - 3 \cos 2y + 3 \cos^2 2y - \cos^3 2y) dy \\ &= \frac{1}{8} \int_0^{\pi} \left[ 1 - 3 \cos 2y + 3 \left( \frac{1}{2} + \frac{1}{2} \cos 4y \right) - (1 - \sin^2 2y) \cos 2y \right] dy \\ &= \frac{1}{8} \left[ \frac{5}{2} y - 2 \sin 2y + \frac{3}{8} \sin 4y + \frac{1}{6} \sin^3 y \right]_0^{\pi} = \frac{5\pi}{16} \end{aligned}$$

$$\text{Combining, } \frac{3\pi}{8} - \frac{5\pi}{16} = \frac{\pi}{16}$$

$$\begin{aligned} 13. \int \frac{\sin^6 \theta}{\cos^2 \theta} d\theta &= \int \frac{\sin^2 \theta (1 - \cos^2 \theta)^2}{\cos^2 \theta} d\theta = \int \tan^2 \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) d\theta \\ &= \int (\sec^2 \theta - 1) d\theta - 2 \int \left( \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta + \frac{1}{4} \int (2 \sin \theta \cos \theta)^2 d\theta \\ &= \tan \theta - 2\theta + \frac{1}{2} \sin 2\theta + \frac{1}{4} \int \left( \frac{1}{2} - \frac{1}{2} \cos 4\theta \right) d\theta \\ &= \tan \theta + \frac{1}{2} \sin 2\theta - \frac{1}{32} \sin 4\theta - \frac{15}{8} \theta + C \end{aligned}$$

$$15. \int_0^{2\pi} \sqrt{\frac{1 - \cos t}{2}} dt = \int_0^{2\pi} \left| \sin \frac{t}{2} \right| dt = \int_0^{2\pi} \sin \frac{t}{2} dt = -2 \cos \frac{t}{2} \Big|_0^{2\pi} = 4$$

Note:  $0 \leq t \leq 2\pi \Rightarrow 0 \leq \frac{t}{2} \leq \pi \Rightarrow \sin \frac{t}{2} \geq 0$  on this interval.

$$\begin{aligned} 17. \int_0^{\frac{\pi}{10}} \sqrt{1 + \cos 5\theta} d\theta &= \sqrt{2} \int_0^{\frac{\pi}{10}} \sqrt{\frac{1 + \cos 5\theta}{2}} d\theta = \sqrt{2} \int_0^{\frac{\pi}{10}} \cos \frac{5\theta}{2} d\theta \\ &= \frac{2\sqrt{2}}{5} \sin \frac{5\theta}{2} \Big|_0^{\frac{\pi}{10}} = \frac{2}{5} \end{aligned}$$

$$\begin{aligned}
 19. \int_0^{\frac{\pi}{2}} \theta \sqrt{1 - \cos \theta} \, d\theta &= \sqrt{2} \int_0^{\frac{\pi}{2}} \theta \sin \frac{\theta}{2} \, d\theta = \sqrt{2} \left[ -2\theta \cos \frac{\theta}{2} + \int_0^{\frac{\pi}{2}} 2 \cos \frac{\theta}{2} \, d\theta \right] \\
 &= \sqrt{2} \left( -2\theta \cos \frac{\theta}{2} + 4 \sin \frac{\theta}{2} \right) \Big|_0^{\frac{\pi}{2}} = 4 - \pi
 \end{aligned}$$

$$\begin{aligned}
 21. \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} \, dx &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sec x| \, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x \, dx \\
 &= 2 \ln |\sec x + \tan x| \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2 \ln(\sqrt{2} + 1)
 \end{aligned}$$

$$23. \int_0^{\pi} \sqrt{1 - \cos^2 \theta} \, d\theta = \int_0^{\pi} |\sin \theta| \, d\theta = -\cos \theta \Big|_0^{\pi} = 2$$

$$\begin{aligned}
 25. \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\cot^2 \theta + 1} \, d\theta &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} |\csc \theta| \, d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc \theta \, d\theta \\
 &= -\ln |\csc \theta - \cot \theta| \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \ln \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = \ln(3 + 2\sqrt{2})
 \end{aligned}$$

$$27. \int_{-\pi}^{\pi} \sqrt{1 - \cos^2 x} \sin x \, dx = 0 \text{ because it is the integral of an odd function over a symmetric interval about 0.}$$

$$29. \int \frac{1}{\sin^4 x} \, dx = \int \csc^4 x \, dx = \int (\cot^2 x + 1) \csc^2 x \, dx = -\frac{1}{3} \cot^3 x - \cot x + C$$

$$31. \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^2 x} \, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x \, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1) \, dx = \tan x - x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2 - \frac{\pi}{2}$$

$$33. \int \frac{\sin^2 t}{\cos t} \, dt = \int \frac{1 - \cos^2 t}{\cos t} \, dt = \int (\sec t - \cos t) \, dt = \ln |\sec t + \tan t| - \sin t + C$$

$$35. \int_0^{\pi} \sqrt{1 + \cos 4x} \, dx = \sqrt{2} \int_0^{\pi} |\cos 2x| \, dx = 4\sqrt{2} \int_0^{\frac{\pi}{4}} \cos 2x \, dx = 2\sqrt{2} \sin 2x \Big|_0^{\frac{\pi}{4}} = 2\sqrt{2}$$

$$\begin{aligned}
 37. \int_0^{\pi} x^2 \sin^2 x \, dx &= \pi \int_0^{\frac{\pi}{2}} x^2 \left( \frac{1}{2} - \frac{1}{2} \cos 2x \right) \, dx = \frac{\pi}{2} \int_0^{\pi} (x^2 - x^2 \cos 2x) \, dx \\
 &= \frac{\pi}{2} \left[ \frac{x^3}{3} - \frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{1}{4} \sin 2x \right]_0^{\pi} = \frac{\pi^4}{6} - \frac{\pi^2}{4}
 \end{aligned}$$