TABLE 7.2
How to integrate odd positive powers of sines and cosines

To evaluate	Write	Substitute
$\int \sin^{2n+1} x\, dx$	$(\sin^2 x)^n \sin x = (1 - \cos^2 x)^n \sin x\, dx$	$u = \cos x$, $du = -\sin x\, dx$
$\int \cos^{2n+1} x\, dx$	$(\cos^2 x)^n \cos x = (1 - \sin^2 x)^n \cos x\, dx$	$u = \sin x$, $du = \cos x\, dx$

PROBLEMS

Evaluate the integrals in Problems 1–22. In evaluating the definite integrals, you may wish to refer to the triangles in Fig. 7.9.

1. $\int \sin^5 x\, dx$

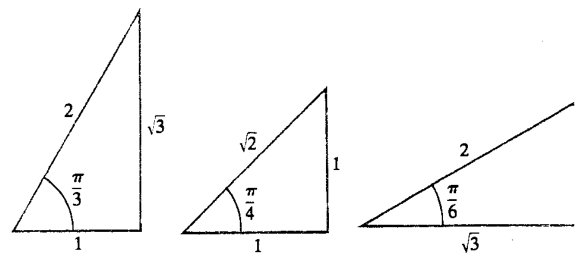
2. $\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$

3. $\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx$

4. $\int \cos^5 3x\, dx$

5. $\int \sin^7 x\, dx$

6. $\int \cos^7 x\, dx$



7.9 Reference triangles for evaluating trigonometric functions at $\pi/6$, $\pi/4$, and $\pi/3$ radians.

7. $\int \cos^{2/3}x \sin^5 x\, dx$

(*Hint:* Put one factor of sin x with dx and express the rest in terms of the cosine.)

8. $\int \sin^{3/2}x \cos^3 x\, dx$

(*Hint:* Put one factor of cos x with dx and express the rest in terms of the sine.)

9. $\int_0^{\pi/3} \sec x\, dx$

10. $\int \sec 4t\, dt$

11. $\int_{-\pi/3}^{0} \sec^3 x\, dx$

12. $\int_0^{\pi/6} \frac{2\sin^2 x}{\cos x}\, dx$

13. $\int e^x \sec^3 e^x\, dx$

14. $\int_0^{\pi/4} \sec^4 x\, dx$

15. $\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^4 x}$

16. $\int \sec^4 3x\, dx$

17. $\int_0^{\pi/4} \sec^8 x\, dx$

18. $\int \frac{2x\, dx}{\cos^3(x^2)}$

19. $\int_0^{\pi/4} \tan^3 x\, dx$

20. $\int_{-\pi/4}^{\pi/4} \tan^4 x\, dx$

21. $\int \tan^6 x\, dx$

22. a) $\int \cot^2 x\, dx$ b) $\int \cot^4 x\, dx$

23. Show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

(*Hint:* Repeat the short derivation of the integral of the secant with cofunctions.)

24. Use the result in Problem 23 to show that

$$\int \csc^3 x\, dx = -\frac{1}{2}\csc x \cot x - \frac{1}{2}\ln|\csc x + \cot x| + C.$$

25. Evaluate

a) $\int_0^{\pi/4} \frac{dx}{\sqrt{1-\sin^2 x}}$, b) $\int_{\pi/3}^{\pi/2} \frac{dx}{\sqrt{1-\cos^2 x}}$.

26. Evaluate

$$\int_0^{\pi/3} \frac{dx}{1+\sin x}.$$

(*Hint:* Multiply numerator and denominator by $1 - \sin x$.)

Evaluate the integrals in Problems 27–32.

27. $\int \sin 3x \cos 2x\, dx$

28. $\int \cos 3x \sin 2x\, dx$

29. $\int_{-\pi}^{\pi} \sin^2 3x\, dx$

30. $\int_0^{\pi/2} \sin x \cos x\, dx$

31. $\int_0^{\pi} \cos 3x \cos 4x\, dx$

32. $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x\, dx$

33. Use the result in Problem 23 and the identity $2 \sin A \cos A = \sin 2A$ to evaluate

$$\int \frac{\sec 2x \csc 2x}{2}\, dx.$$

34. Which integrals are zero and which are not? (You can do most of these without writing anything down.)

a) $\int_{-\pi}^{\pi} \sin x \cos^2 x\, dx$ b) $\int_{-1}^{1} \sin 3x \cos 5x\, dx$

c) $\int_{-L}^{L} \sqrt[3]{\sin x}\, dx$ d) $\int_{-a}^{a} x\sqrt{a^2 - x^2}\, dx$

e) $\int_{-\pi/4}^{\pi/4} x \sec x\, dx$ f) $\int_{-\pi/4}^{\pi/4} \tan^3 x\, dx$

g) $\int_{-\pi/2}^{\pi/2} x \sin x\, dx$ h) $\int_{-\pi/2}^{\pi/2} x \cos x\, dx$

i) $\int_{-a}^{a} \sin mx \cos mx\, dx,\ m \neq 0$ j) $\int_{-\pi}^{\pi} \sin^5 x\, dx$

k) $\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx$ l) $\int_{-\pi}^{\pi} \cos^5 x\, dx$

m) $\int_{-\ln 2}^{\ln 2} x(e^x + e^{-x})\, dx$ n) $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$

o) $\int_{-\pi/2}^{\pi/2} \sin x \sin 2x\, dx$ p) $\int_{-\pi/4}^{\pi/4} \sec x \tan x\, dx$

q) $\int_{-a}^{a} (e^x \sin x + e^{-x}\sin x)\, dx$ r) $\int_{-1}^{1} \frac{\sin x\, dx}{e^x + e^{-x}}$

35. Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

36. Find the length of the curve

$$y = \ln(\cos x), \quad 0 \le x \le \pi/3.$$

37. Find the length of the curve

$$y = \ln(\sec x), \quad 0 \le x \le \pi/4.$$

38. Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

39. **CALCULATOR OR TABLES** How far apart should the following lines of latitude be on the Mercator map of Example 7?

a) 30° and 45° north (about the latitudes of New Orleans, La., and Minneapolis, Minn.);

b) 45° and 60° north (about the latitudes of Salem, Ore., and Seward, Al.).

40. Show that

a) $\int_0^{2\pi} \sin mx \sin nx\, dx = 0,$

41. (a) $\int_k^{k+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_k^{k+2\pi} [\cos(m-n)x - \cos(m+n)x]\, dx$

$= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)},\ m^2 \neq n^2 \Big]_k^{k+2\pi} = 0$

since since $\sin k = \sin(k+2\pi)$

(b) $\int_k^{k+2\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_k^{k+2\pi} [\cos(m-n)x + \cos(m+n)x]\, dx$

$= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)},\ m^2 \neq n^2 \Big]_k^{k+2\pi} = 0$ as in part (a).

7.4 EVEN POWERS OF SINES AND COSINES

1. $\int_{-\pi}^{\pi} \sin^2 x\, dx = \int_{-\pi}^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x - \frac{1}{4}\sin 2x\Big]_{-\pi}^{\pi} = \pi$

3. $\int \sin^2 2t\, dt = \int\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right) dt = \frac{1}{2}t - \frac{1}{8}\sin 4t + C$

5. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x \cos^2 x\, dx = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\sin x \cos x)^2\, dx = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 2x\, dx$

$= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx = \frac{1}{8}x - \frac{1}{32}\sin 4x\Big]_{\frac{-\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{16}$

7. $\int_0^{\frac{\pi}{a}} \sin^4 ax\, dx = \int_0^{\frac{\pi}{a}} (\sin^2 ax)^2\, dx = \int_0^{\frac{\pi}{a}}\left(\frac{1}{2} - \frac{1}{2}\cos 2ax\right)^2 dx$

$= \int_0^{\frac{\pi}{a}}\left(\frac{1}{4} - \frac{1}{2}\cos 2ax + \frac{1}{4}\cos^2 2ax\right) dx$

$= \int_0^{\frac{\pi}{a}}\left[\left(\frac{1}{4} - \frac{1}{2}\cos 2ax + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4ax\right)\right)\right] dx$

$= \frac{3}{8}x - \frac{1}{4a}\sin 2ax + \frac{1}{32a}\sin 4ax\Big]_0^{\frac{\pi}{a}} = \frac{3\pi}{8a}$

9. $\int \frac{\sin^4 x}{\cos^2 x}\, dx = \int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}\, dx = \int (\tan^2 x - \sin^2 x)\, dx$

$= \int\left(\sec^2 x - 1 - \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\right) dx = \tan x - \frac{3}{2}x + \frac{1}{4}\sin 2x + C$

11. $\int_0^{\pi} \sin^4 y \cos^2 y\, dy = \int_0^{\pi} \sin^4 y (1 - \sin^2 y)\, dy = I_1 - I_2$, where

$$I_1 = \int_0^{\pi} \sin^4 y\, dy = \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2y\right)^2 dy$$

$$= \int_0^{\pi} \left(\frac{1}{4} - \frac{1}{2}\cos 2y + \frac{1}{4}\cos^2 2y\right) dy$$

$$= \int_0^{\pi} \left(\frac{1}{4} - \frac{1}{2}\cos 2y + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4y\right)\right) dy$$

$$= \frac{3}{8} y - \frac{1}{4}\sin 2y + \frac{1}{32}\sin 4y \Big]_0^{\pi} = \frac{3\pi}{8}$$

and $I_2 = \int_0^{\pi} \sin^6 y\, dy = \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2y\right)^3 dy$

$$= \frac{1}{8}\int_0^{\pi} (1 - 3\cos 2y + 3\cos^2 2y - \cos^3 2y)\, dy$$

$$= \frac{1}{8}\int_0^{\pi} \left[1 - 3\cos 2y + 3\left(\frac{1}{2} + \frac{1}{2}\cos 4y\right) - (1 - \sin^2 2y)\cos 2y\right] dy$$

$$= \frac{1}{8}\left[\frac{5}{2}y - 2\sin 2y + \frac{3}{8}\sin 4y + \frac{1}{6}\sin^3 y\right]_0^{\pi} = \frac{5\pi}{16}$$

Combining, $\frac{3\pi}{8} - \frac{5\pi}{16} = \frac{\pi}{16}$

13. $\int \frac{\sin^6\theta}{\cos^2\theta}\, d\theta = \int \frac{\sin^2\theta\,(1 - \cos^2\theta)^2}{\cos^2\theta}\, d\theta = \int \tan^2\theta\,(1 - 2\cos^2\theta + \cos^4\theta)\, d\theta$

$$= \int (\sec^2\theta - 1)\, d\theta - 2\int \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta + \frac{1}{4}\int (2\sin\theta\cos\theta)^2\, d\theta$$

$$= \tan\theta - 2\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \tan\theta + \frac{1}{2}\sin 2\theta - \frac{1}{32}\sin 4\theta - \frac{15}{8}\theta + C$$

15. $\int_0^{2\pi} \sqrt{\frac{1 - \cos t}{2}}\, dt = \int_0^{2\pi} \left|\sin\frac{t}{2}\right| dt = \int_0^{2\pi} \sin\frac{t}{2}\, dt = -2\cos\frac{t}{2}\Big]_0^{2\pi} = 4$

Note: $0 \le t \le 2\pi \Rightarrow 0 \le \frac{t}{2} \le \pi \Rightarrow \sin\frac{t}{2} \ge 0$ on this interval.

17. $\int_0^{\frac{\pi}{10}} \sqrt{1 + \cos 5\theta}\, d\theta = \sqrt{2}\int_0^{\frac{\pi}{10}} \sqrt{\frac{1 + \cos 5\theta}{2}}\, d\theta = \sqrt{2}\int_0^{\frac{\pi}{10}} \cos\frac{5\theta}{2}\, d\theta$

$$= \frac{2\sqrt{2}}{5}\sin\frac{5\theta}{2}\Big]_0^{\frac{\pi}{10}} = \frac{2}{5}$$

19. $\int_0^{\frac{\pi}{2}} \theta\sqrt{1-\cos\theta}\, d\theta = \sqrt{2}\int_0^{\frac{\pi}{2}} \theta \sin\frac{\theta}{2}\, d\theta = \sqrt{2}\left[-2\theta\cos\frac{\theta}{2} + \int_0^{\frac{\pi}{2}} 2\cos\frac{\theta}{2}\, d\theta\right]$

$= \sqrt{2}\left(-2\theta\cos\frac{\theta}{2} + 4\sin\frac{\theta}{2}\Big]_0^{\frac{\pi}{2}}\right) = 4 - \pi$

21. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sec x|\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x\, dx$

$= 2\ln|\sec x + \tan x|\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2\ln(\sqrt{2}+1)$

23. $\int_0^{\pi} \sqrt{1-\cos^2\theta}\, d\theta = \int_0^{\pi} |\sin\theta|\, d\theta = -\cos\theta\Big]_0^{\pi} = 2$

25. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{\cot^2\theta + 1}\, d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} |\csc\theta|\, d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc\theta\, d\theta$

$= -\ln|\csc\theta - \cot\theta|\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1} = \ln(3+2\sqrt{2})$

27. $\int_{-\pi}^{\pi} \sqrt{1-\cos^2 x}\sin x\, dx = 0$ because it the integral of an odd function over a symmetric interval about 0.

29. $\int \frac{1}{\sin^4 x}\, dx = \int \csc^4 dx = \int (\cot^2 x + 1)\csc^2 x\, dx = -\frac{1}{3}\cot^3 x - \cot x + C$

31. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^2 x}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1)\, dx = \tan x - x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2 - \frac{\pi}{2}$

33. $\int \frac{\sin^2 t}{\cos t}\, dt = \int \frac{1-\cos^2 t}{\cos t}\, dt = \int (\sec t - \cos t)\, dt = \ln|\sec t + \tan t| - \sin t + C$

35. $\int_0^{\pi} \sqrt{1+\cos 4x}\, dx = \sqrt{2}\int_0^{\pi} |\cos 2x|\, dx = 4\sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\, dx = 2\sqrt{2}\sin 2x\Big]_0^{\frac{\pi}{4}} = 2\sqrt{2}$

37. $V = \pi\int_0^{\pi} x^2\sin^2 x\, dx = \pi\int_0^{\pi} x^2\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx = \frac{\pi}{2}\int_0^{\pi} (x^2 - x^2\cos 2x)\, dx$

$= \frac{\pi}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi^4}{6} - \frac{\pi^2}{4}$