Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse of the natural logarithm function $y = \ln x$; that is, $e^x = \ln^{-1}x$.

   Domain: The set of all real numbers, $-\infty < x < \infty$.

   Range: The set of all positive numbers, $y > 0$.

2. Its derivative is

$$\frac{d}{dx}(e^x) = e^x.$$

3. It is continuous (because it is differentiable) and is an increasing function of x.
4. If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \quad \text{and} \quad \int e^u\,du = e^u + C.$$

5. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$ and $e^{-x} = 1/e^x$

PROBLEMS

Simplify the expressions in Problems 1–12.

1. $e^{\ln x}$ **2.** $\ln(e^x)$ **3.** $e^{-\ln(x^2)}$

4. $\ln(e^{-x^2})$ **5.** $\ln(e^{1/x})$ **6.** $\ln(1/e^x)$

7. $e^{\ln 2+\ln x}$ **8.** $e^{2\ln x}$ **9.** $\ln(e^{x-x^2})$

10. $\ln(x^2e^{-2x})$ **11.** $e^{x+\ln x}$ **12.** $e^{\ln x-2\ln y}$

In Problems 13–18, solve for y.

13. $e^{\sqrt{y}} = x^2$

14. $e^{2y} = x^2$

15. $e^{(x^2)} \cdot e^{(2x+1)} = e^y$

16. $\ln(y-1) = x + \ln x$

17. $\ln(y-2) = \ln(\sin x) - x$

18. $\ln(y^2-1) - \ln(y+1) = \sin x$

Find dy/dx in Problems 19–42.

19. $y = e^{3x}$ **20.** $y = e^{(x+1)}$

21. $y = e^{5-7x}$ **22.** $y = \cos e^x$

23. $y = x^2e^x$ **24.** $y = \sin e^{-x}$

25. $y = e^{\sin x}$ **26.** $y = e^{(x^2)} \cdot e^{-x}$

27. $y = \ln(3xe^{-x})$ **28.** $y = \ln\dfrac{e^x}{1+e^x}$

29. $y = e^{\sin^{-1}x}$ **30.** $y = (1+2x)e^{-2x}$

31. $y = (9x^2 - 6x + 2)e^{3x}$ **32.** $y = \dfrac{ax-1}{a^2}e^{ax}$

33. $y = x^2e^{-x^2}$ **34.** $y = e^x \ln x$

35. $y = \tan^{-1}(e^x)$ **36.** $y = \sec^{-1}(e^{2x})$

37. $y = x^3e^{-2x}\cos 5x$ (*Hint:* Use logarithmic differentiation.)

38. $y = \displaystyle\int_0^{\ln x} \sin e^t\,dt, \quad x > 0$

39. $\ln y = x \sin x$ **40.** $\ln xy = e^{x+y}$

41. $e^{2x} = \sin(x+3y)$ **42.** $\tan y = e^x + \ln x$

Evaluate the integrals in Problems 43–56.

43. $\displaystyle\int_{\ln 3}^{\ln 5} e^{2x}\,dx$ **44.** $\displaystyle\int_{-1}^{1} xe^{x^2}\,dx$

45. $\displaystyle\int_0^{\pi} e^{\sin x}\cos x\,dx$ **46.** $\displaystyle\int_0^{\ln 8} e^{x/3}\,dx$

47. $\displaystyle\int_{-\ln(a+1)}^{0} e^{-x}\,dx$ **48.** $\displaystyle\int_0^2 e^{x/2}\,dx$

49. $\displaystyle\int_0^1 e^{\ln\sqrt{x}}\,dx$ **50.** $\displaystyle\int_0^1 \frac{dx}{e^x}$

51. $\displaystyle\int_0^{\ln 2} \frac{24\,dx}{e^{3x}}$ **52.** $\displaystyle\int_0^1 \frac{e^x\,dx}{1+e^x}$

53. $\displaystyle\int_0^{\ln 13} \frac{e^x\,dx}{1+2e^x}$ **54.** $\displaystyle\int_e^{e^2} \frac{dx}{x\ln x}$

55. $\displaystyle\int_0^{\ln 2} \frac{e^x\,dx}{1+e^{2x}}$ (*Hint:* Let $u = e^x$.)

56. $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$ (*Hint:* Let $u = \sqrt{x}$.)

gives a way to define e independently of the definition of the natural logarithm. However, the proof that this limit exists would then have to be different from the proof here which uses the logarithm.

Properties of $y = a^x$, $a > 0$, $a \neq 1$

If a is a positive real number and $a \neq 1$, then the function $y = a^x$ has the following properties.

1. It is defined by the equation

$$a^x = e^{x \ln a}.$$

Domain: the set of all real numbers, $-\infty < x < \infty$.

Range: the set of all positive real numbers, $y > 0$.

2. Its derivative is

$$\frac{d}{dx}a^x = a^x \ln a.$$

3. It is continuous (because it is differentiable), increasing if $a > 1$, decreasing if $0 < a < 1$, and one-to-one in either case.
4. If u is any differentiable function of x, then

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx} \quad \text{and} \quad \int a^u\, du = \frac{1}{\ln a}a^u + C.$$

PROBLEMS

In Problems 1–16, find dy/dx.

1. $y = 2^x$
2. $y = 2^{3x}$
3. $y = 8^x$
4. $y = 3^{2x}$
5. $y = 9^x$
6. $y = 2^x 3^x$
7. $y = (2^x)^2$
8. $y = x^{\sin x}$, $x > 0$
9. $y = (\sin x)^{\tan x}$, $\sin x > 0$
10. $y = 2^{\sec x}$
11. $y = x^{\ln x}$, $x > 0$
12. $y = (\cos x)^x$, $\cos x > 0$
13. $y = (1 - x)^x$, $x < 1$
14. $y = x2^{(x^2)}$
15. $y = 2^x \ln x$
16. $y = (\cos x)^{\sqrt{x}}$, $x > 0$, $\cos x > 0$

Evaluate the integrals in Problems 17–28.

17. $\int 5^x\, dx$
18. $\int_{-1}^{0} 2^x\, dx$
19. $\int \frac{1}{2^x} dx$
20. $\int_{-1}^{1} \left(\frac{1}{10}\right)^x dx$
21. $\int_0^1 3^{2x}\, dx$
22. $\int_{-1}^{1} 2^{(x+1)}\, dx$
23. $\int_{-1}^{0} 4^{-x} \ln 2\, dx$
24. $\int_{-2}^{0} 5^{-x}\, dx$
25. $\int_1^2 5^{(2x-2)}\, dx$
26. $\int_1^{\sqrt{2}} x2^{-x^2}\, dx$
27. $\int_0^{\pi/2} 2^{\cos x} \sin x\, dx$
28. $\int_0^{\pi/3} 2^{\sec x} \sec x \tan x\, dx$
29. Which integral has the larger value: a, or b?

a) $\int_0^1 2^{(3x)}\, dx$ b) $\int_0^1 3^{(2x)}\, dx$

30. Find the derivative with respect to x of the following functions.

a) $y = 2^{\ln x}$ b) $y = \ln 2^x$
c) $y = \ln x^2$ d) $y = (\ln x)^2$

Find the limits in Problems 31–36.

31. $\lim_{x\to\infty} 2^{-x}$
32. $\lim_{x\to-\infty} 3^x$

6.6 THE EXPONENTIAL FUNCTION e^x

1. $e^{\ln x} = x$

3. $e^{-\ln (x^2)} = e^{\ln (x^{-2})} = x^{-2}$

5. $\ln e^{\frac{1}{x}} = \frac{1}{x}$

7. $e^{\ln 2 + \ln x} = e^{\ln 2x} = 2x$

9. $\ln e^{(x-x^2)} = x - x^2$

11. $e^{(x+\ln x)} = e^x e^{\ln x} = xe^x$

13. $e^{\sqrt{y}} = x^2 \Rightarrow \sqrt{y} \ln e = \ln x^2 \Rightarrow \sqrt{y} = 2 \ln x \Rightarrow y = 4\ln^2 x$

15. $e^{(x^2)} e^{2x+1} = e^y \Leftrightarrow y = x^2 + 2x + 1$

17. $\ln (y-2) = \ln (\sin x) - x \Rightarrow \ln\frac{y-2}{\sin x} = -x \Rightarrow \frac{y-2}{\sin x} = e^{-x}$

    $y = e^{-x} \sin x + 2.$

19. $y = e^{3x} \Rightarrow \frac{dy}{dx} = 3e^{3x}$

21. $y = e^{5-7x} \Rightarrow \frac{dy}{dx} = -7e^{5-7x}$

23. $y = x^2 e^x = \frac{dy}{dx} = x^2 e^x + 2xe^x = xe^x (x+2)$

25. $y = e^{\sin x} \Rightarrow \frac{dy}{dx} = (\cos x)e^{\sin x}$

27. $y = \ln(3xe^{-x}) = \ln 3 + \ln x - x \Rightarrow \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1-x}{x}$

29. $y = e^{\sin^{-1} x} \Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x}\left(\frac{1}{\sqrt{1-x^2}}\right)$

31. $y = (9x^2 - 6x + 2)e^{3x} \Rightarrow \frac{dy}{dx} = 3(9x^2 - 6x + 2)e^{3x} + (18x - 6)e^{3x} = 27x^2 e^{3x}$

33. $y = x^2 e^{-(x^2)} \Rightarrow \frac{dy}{dx} = x^2(-2xe^{-(x^2)}) + 2xe^{-(x^2)} = 2xe^{-(x^2)}(1 - x^2)$

35. $y = \tan^{-1}(e^x) \Rightarrow \frac{dy}{dx} = \frac{e^x}{1 + e^{2x}}$

37. $y = x^3 e^{-2x} \cos 5x \Rightarrow \ln y = 3\ln x - 2x + \ln(\cos 5x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - 2 - \frac{5\sin 5x}{\cos 5x} = \frac{3}{x} - 2 - 5\tan 5x$$

$$\frac{dy}{dx} = y\left(\frac{3}{x} - 2 - 5\tan 5x\right)$$

39. $\ln y = x\sin x \Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x + x\cos x \Rightarrow \frac{dy}{dx} = y(\sin x + x\cos x)$

41. $e^{2x} = \sin(x + 3y) \Rightarrow 2e^{2x} = \cos(x+3y)\left[1 + 3\frac{dy}{dx}\right]$

$$2e^{2x} = \cos(x+3y) + 3\cos(x+3y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2e^{2x} - \cos(x+3y)}{3\cos(x+3y)}$$

43. $\int_{\ln 3}^{\ln 5} e^{2x}\,dx = \frac{1}{2}e^{2x}\Big]_{\ln 3}^{\ln 5} = \frac{1}{2}[e^{2\ln 5} - e^{2\ln 3}] = \frac{1}{2}(25 - 9) = 8$

45. $\int_0^{\pi} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi} = 0$

47. $\int_{-\ln(a+1)}^{0} e^{-x}\,dx = -e^{-x}\Big]_{-\ln(a+1)}^{0} = -[e^0 - e^{\ln(a+1)}] = a$

49. $\int_0^1 e^{\ln\sqrt{x}}\,dx = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$

51. $\int_0^{\ln 2} \frac{24\,dx}{e^{3x}} = -8\,e^{-3x}\Big]_0^{\ln 2} = -8\,(e^{-3\ln 2} - e^0) = -8\left(\frac{1}{8} - 1\right) = 7$

53. $\int_0^{\ln 13} \frac{e^x\,dx}{1+2e^x} = \frac{1}{2}\ln(1+2e^x)\Big]_0^{\ln 13} = \frac{1}{2}(\ln 27 - \ln 3) = \ln 3$

55. $\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\,dx = \tan^{-1} e^x\Big]_0^{\ln 2} = \tan^{-1} 2 - \frac{\pi}{4}$

57. $\lim_{h\to 0}\frac{e^h-(1+h)}{h^2} = \lim_{h\to 0}\frac{e^h-1}{2h} = \lim_{h\to 0}\frac{e^h}{2} = \frac{1}{2}$

59. $\lim_{x\to\infty}\frac{x^2+e^x}{x+e^x} = \lim_{x\to\infty}\frac{2x+e^x}{1+e^x} = \lim_{x\to\infty}\frac{2+e^x}{e^x} = \lim_{x\to\infty}\frac{e^x}{e^x} = 1$

61. (a) $y = e^{\sin x}$, $-\pi \le x \le 2\pi$. $y'(x) = (\cos x)e^{\sin x} = 0 \Leftrightarrow \cos x = 0$.

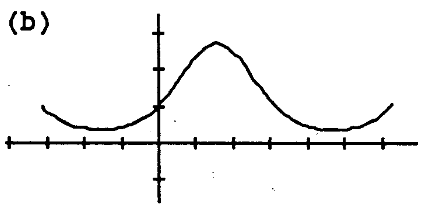
$\therefore x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. $y\left(\frac{\pi}{2}\right) = e^{\sin\left(\frac{\pi}{2}\right)} = e$. $y\left(-\frac{\pi}{2}\right) = y\left(\frac{3\pi}{2}\right) = \frac{1}{e}$.

Testing endpoints: $y(\pi) = e^{\sin\pi} = 1$ and $y(2\pi) = 1$.

The absolute maximum $= e$ at $x = \frac{\pi}{2}$ and the

absolute minimum $= \frac{1}{e}$ at $x = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

(b)



63. $f(x) = x^2\ln\frac{1}{x} = -x^2\ln x \Rightarrow f'(x) = x^2\left(\frac{1}{x}\right) - 2x\ln x = 0$ if

$-x(1+2\ln x) = 0 \Leftrightarrow x = 0$ or $x = e^{-1/2}$. $f''(x) = -3 - 2\ln x$

and $f''(e^{-1/2}) = -2 \Rightarrow f(e^{-1/2}) = \frac{1}{2e}$ is a maximum.

65. (a) $f(x) = f'(x) = f''(x) = f'''(x) = e^x$

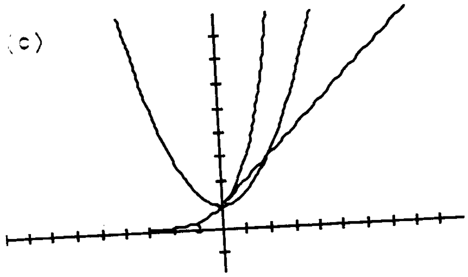
Linear: $e^x \approx f(0) + f'(0)x = 1 + x$

Quadratic: $e^x \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + x + \frac{1}{2}x^2$

(b) $|e_1(x)| \le \frac{1}{2}Mx^2 = \frac{1}{2}(1)(0.1)^2 = 0.005$

$|e_2(x)| \le \frac{1}{6}Mx^3 = \frac{1}{6}(1)(0.1)^3 \approx 0.00017$

(c)



67. $A(t) = \int_0^t e^{-x}\,dx = -e^{-x}\Big]_0^t = -e^{-t} + 1$

$V(t) = \pi\int_0^t (e^{-x})^2\,dx = \pi\left[-\frac{1}{2}e^{-2x}\right]_0^t = \frac{\pi}{2}(1 - e^{-2t})$

(a) $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} (-e^{-t} + 1) = 1$

(b) $\lim_{t\to\infty} \frac{V(t)}{A(t)} = \lim_{t\to\infty} \frac{\pi}{2}\left(\frac{1 - e^{-2t}}{-e^{-t} + 1}\right) = \frac{\pi}{2}$

(c) $\lim_{t\to 0^+} \frac{V(t)}{A(t)} = \lim_{t\to 0^+} \frac{\pi}{2}\left(\frac{1 - e^{-2t}}{-e^{-t} + 1}\right) = \lim_{t\to 0^+} \frac{\pi}{2}\left(\frac{2e^{-2t}}{e^{-t}}\right) = \pi$

69. (a) $y = Ce^{ax} \Rightarrow \frac{dy}{dx} = aCe^{ax} = ay$

(b) $\frac{dy}{dt} = -2y \Rightarrow y = Ce^{-2t}$. $3 = Ce^{0} \Rightarrow C = 3$. $\therefore y = 3e^{-2t}$

71. $\frac{dy}{dx} = e^{-x}$, $y = 0$ when $x = 4 \Rightarrow y = -e^{-x} + C$.

$0 = -e^{-4} + C \Rightarrow C = e^{-4}$. $\therefore\ y = -e^{-x} + e^{-4}$

73. $\frac{1}{y+1}\frac{dy}{dx} = \frac{1}{2x}$, $x > 0$, $y = 1$ when $x = 2$

$\frac{dy}{y+1} = \frac{dx}{2x} \Rightarrow \ln|y+1| = \frac{1}{2}\ln|x| + C$

$C = \ln 2 - \ln\sqrt{2} = \ln\frac{2}{\sqrt{2}} = \ln\sqrt{2}$

$\ln|1+y| = \ln\sqrt{x} + \ln\sqrt{2} = \ln\sqrt{2x} \Rightarrow y + 1 = \sqrt{2x}$ or $(y+1)^2 = 2x$

75. $\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left[\frac{1}{2}(e^{x} + e^{-x})\right] = \frac{1}{2}(e^{x} - e^{-x}) = \sinh x$

77. $\cosh(-x) = \frac{1}{2}(e^{-x} + e^{x}) = \frac{1}{2}(e^{x} - e^{-x}) = \cosh x$

$\sinh(-x) = \frac{1}{2}(e^{-x} - e^{x}) = -\frac{1}{2}(e^{x} - e^{-x}) = -\sinh x$

6.7 THE FUNCTIONS a^x and a^u

1. If $y = 2^x = e^{\ln 2^x} = e^{x \ln 2}$, then $\frac{dy}{dx} = (\ln 2)\, e^{x \ln 2} = (\ln 2)2^x$

   or $y = 2^x \Rightarrow \ln y = \ln 2^x = x \ln 2$, so $\frac{1}{y}\frac{dy}{dx} = \ln 2$, $\frac{dy}{dx} = y \ln 2 = (\ln 2)2^x$

3. $y = 8^x = e^{\ln 8^x} = e^{x \ln 8} \Rightarrow \frac{dy}{dx} = (\ln 8)\, e^{x \ln 8} = (\ln 8)8^x$

5. $y = 9^x = e^{\ln 9^x} = e^{x \ln 9} \Rightarrow \frac{dy}{dx} = (\ln 9)\, e^{x \ln 9} = (\ln 9)9^x$

7. $y = (2^x)^2 = (2^2)^x = 4^x.\ \frac{dy}{dx} = (\ln 4)4^x$

9. $y = (\sin x)^{\tan x}$, $\sin x > 0 \Rightarrow \ln y = (\tan x)\ln(\sin x)$

   $\frac{1}{y}\frac{dy}{dx} = (\tan x)\left(\frac{\cos x}{\sin x}\right) + [\ln(\sin x)](\sec^2 x)$

   $\frac{dy}{dx} = y\,[1 + (\sec^2 x)\ln(\sin x)]$

11. $y = x^{\ln x}$, $x > 0 \Rightarrow \ln y = (\ln x)(\ln x) = \ln^2 x$

    $\frac{1}{y}\frac{dy}{dx} = 2 \ln x\left(\frac{1}{x}\right) \Rightarrow \frac{dy}{dx} = 2y\left(\frac{\ln x}{x}\right) = \frac{2 \ln x\,(x^{\ln x})}{x}$

13. $y = (1-x)^x$, $x < 1 \Rightarrow \ln y = x \ln(1-x)$

    $\frac{1}{y}\frac{dy}{dx} = \frac{-x}{1-x} + \ln(1-x) \Rightarrow \frac{dy}{dx} = y\left(\frac{x}{x-1} + \ln(1-x)\right)$ or

    $\frac{dy}{dx} = (1-x)^x\left(\frac{x}{x-1} + \ln(1-x)\right)$

15. $y = 2^x \ln x \Rightarrow \frac{dy}{dx} = 2^x\left(\frac{1}{x}\right) + \ln x\,(\ln 2)2^x = 2^x\left[\frac{1}{x} + \ln 2\,(\ln x)\right]$

17. $\int_0^1 5^x\,dx = \frac{1}{\ln 5}5^x\Big]_0^1 = \frac{1}{\ln 5}(5-1) = \frac{4}{\ln 5}$

19. $\int_0^1 \frac{1}{2^x}dx = \int_0^1 2^{-x}dx = -\frac{1}{\ln 2}2^{-x}\Big]_0^1 = -\frac{1}{\ln 2}\left[\frac{1}{2} - 1\right] = \frac{1}{2 \ln 2}$

21. $\int_0^1 3^{2x}\,dx = \frac{1}{2\ln 3}3^{2x}\Big]_0^1 = \frac{1}{2 \ln 3}[3^2 - 1] = \frac{4}{\ln 3}$

23. $\int_{-1}^0 4^{-x} \ln 2\,dx = -\frac{\ln 2}{\ln 4}4^{-x}\Big]_{-1}^0 = -\frac{\ln 2}{\ln 4}[1-4] = \frac{3 \ln 2}{\ln 4}$

25. $\int_1^2 5^{(2x-2)}\,dx = \frac{1}{2 \ln 5}5^{(2x-2)}\Big]_1^2 = \frac{1}{2 \ln 5}[5^2 - 5^0] = \frac{24}{\ln 25}$

27. $\int_0^{\frac{\pi}{2}} 2^{\cos x}\sin x\,dx = \frac{-1}{\ln 2}2^{\cos x}\Big]_0^{\frac{\pi}{2}} = -\frac{1}{\ln 2}[2^0 - 2] = \frac{1}{\ln 2}$

29. (b), because $\int_0^1 2^{3x}\,dx = \int_0^1 8^x\,dx$, and $\int_0^1 3^{2x}\,dx = \int_0^1 9^x\,dx$

31. $\lim_{x\to\infty} 2^{-x} = 0$

33. $\lim_{x\to 0} \frac{3^{\sin x} - 1}{x} = \lim_{x\to 0} \frac{3^{\sin x}(\cos x)(\ln 3)}{1} = \ln 3$

35. Let $y = (e^x + x)^{\frac{1}{x}}$ so that $\ln y = \frac{1}{x}\ln(e^x + x)$. We first find

$\lim_{x\to 0} \frac{\ln(e^x + x)}{x} = \lim_{x\to 0} \frac{e^x + 1}{e^x + x} = 2$. Thus $\lim_{x\to 0} \ln y = 2$, so

$\lim_{x\to 0} e^{\ln y} = \lim_{x\to 0} (e^x + x)^{\frac{1}{x}} = e^2$.

37. (a) $\lim_{x\to\infty} \frac{3^x - 5}{4(3^x + 2)} = \lim_{x\to\infty} \frac{1 - \frac{5}{3^x}}{4\left(1 + \frac{2}{3^x}\right)} = \frac{1}{4}$

(b) $\lim_{x\to -\infty} \frac{3^x - 5}{4(3^x + 2)} = \frac{0 - 5}{4(0 + 2)} = -\frac{5}{8}$

39. Let $f(x) = 2^x - x^2$, so that $f'(x) = (\ln 2)\,2^x - 2x$. Let $x_0 = -0.5$.

Using the iterative formula $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$, we find

$x = -0.766664696$, $y = 0.587774756$.

41. Let $y = x^{(1/x^n)}$ and consider $\ln y = \frac{1}{x^n}\ln x$.

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x^n} = \lim_{n\to\infty} \frac{\frac{1}{x}}{nx^{n-1}} = \lim_{n\to\infty} \frac{1}{nx^n} = 0$. Hence

$\lim_{x\to\infty} e^{\ln y} = \lim_{x\to\infty} x^{(1/x^n)} = e^0 = 1$

43. Equation 2 implies line (a), commutativity of multiplication implies line (b), Equaion 4 implies line (c), and Equation 2 implies line (d).