Infinity as a limit:

(Compare with Example 5.)

$$\lim_{x\to\infty} \frac{2x^2 - x + 3}{3x^2 + 5} = \lim_{h\to 0^+} \frac{(2/h^2) - (1/h) + 3}{(3/h^2) + 5}$$

$$= \lim_{h\to 0^+} \frac{2 - h + 3h^2}{3 + 5h^2} = \frac{2 - 0 + 3(0)^2}{3 + 5(0)^2} = \frac{2}{3}$$

$$\lim_{x\to\infty} \frac{5x + 3}{2x^2 - 1} = \lim_{h\to 0^+} \frac{(5/h) + 3}{(2/h^2) - 1} = \lim_{h\to 0^+} \frac{5h + 3h^2}{2 - h^2} = \frac{0}{2} = 0$$

To calculate the limit of a quotient of two polynomials as $x \to -\infty$, we may substitute $h = 1/x$ and calculate the limit as $h \to 0^-$.

(Compare with Example 9.)

$$\lim_{x\to -\infty} \frac{2x^2 - 3}{7x + 4} = \lim_{h\to 0^-} \frac{(2/h^2) - 3}{(7/h) + 4} = \lim_{h\to 0^-} \frac{2 - 3h^2}{7h + 4h^2} = -\infty$$

The substitution $x = 1/h$ may help in calculating limits of other functions as well.

$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{h\to 0^+} \frac{\sin h}{h} = 1$$

Find the limits in Problems 1–32.

1. $\lim_{x\to\infty} \frac{2x + 3}{5x + 7}$
2. $\lim_{t\to\infty} \frac{t^3 + 7}{t^4}$
3. $\lim_{x\to\infty} \frac{x + 1}{x^2 + 3}$
4. $\lim_{x\to\infty} \frac{3x^2 - 6x}{4x - 8}$
5. $\lim_{y\to\infty} \frac{3y + 7}{y^2 - 2}$
6. $\lim_{x\to\infty} \frac{7x - 28}{x^3}$
7. $\lim_{t\to\infty} \frac{t^2 - 2t + 3}{2t^2 + 5t - 3}$
8. $\lim_{t\to\infty} \frac{t^2 + 1}{t + 1}$
9. $\lim_{x\to\infty} \frac{x}{x - 1}$
10. $\lim_{x\to\infty} [x]$
11. $\lim_{x\to -\infty} |x|$
12. $\lim_{x\to -\infty} \frac{1}{|x|}$
13. $\lim_{a\to\infty} \frac{|a|}{|a| + 1}$
14. $\lim_{t\to -\infty} \frac{t}{t + 1}$
15. $\lim_{x\to\infty} \frac{3x^3 + 5x^2 - 7}{10x^3 - 11x + 5}$
16. $\lim_{x\to\infty} \left(\frac{1}{x} + 1\right)\left(\frac{5x^2 - 1}{x^2}\right)$
17. $\lim_{s\to\infty} \left(\frac{s}{s + 1}\right)\left(\frac{s^2}{5 + s^2}\right)$
18. $\lim_{x\to\infty} \frac{8x^{23} - 7x^2 + 5}{2x^{23} + x^{22}}$
19. $\lim_{r\to -\infty} \frac{8r^2 + 7r}{4r^2}$
20. $\lim_{x\to\infty} \frac{7x^3}{x^3 - 3x^2 + 6x}$
21. $\lim_{y\to\infty} \frac{y^4}{y^4 - 7y^3 + 7y^2 + 9}$
22. $\lim_{x\to\infty} \frac{5x^3 - 6x + 2}{10x^3 + 5}$
23. $\lim_{x\to\infty} \frac{x - 3}{x^2 - 5x + 4}$
24. $\lim_{x\to\infty} \frac{9x^4 + x}{2x^4 + 4x^2 - x + 6}$
25. $\lim_{x\to\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$
26. $\lim_{x\to -\infty} \frac{-2x^3 - 2x + 3}{3x^3 + 3x^2 - 5x}$
27. $\lim_{x\to\infty} \frac{x + \sin x}{x + \cos x}$
28. $\lim_{x\to -\infty} \frac{1 - x^2}{1 + 2x^2}$
29. $\lim_{x\to\infty} \left(\frac{1}{x^4} + \frac{1}{x}\right)$
30. $\lim_{x\to\infty} \frac{\sin 2x}{x}$
31. $\lim_{x\to\infty} \left(1 + \cos\frac{1}{x}\right)$
32. $\lim_{y\to\infty} \frac{1}{y^2 + 5}$

Find the limits in Problems 33–50.

33. $\lim_{x\to 0^+} \frac{1}{3x}$
34. $\lim_{x\to 0^-} \frac{2}{x}$

35. $\lim_{x\to 0^+} \dfrac{5}{2x}$

36. $\lim_{t\to 2^+} \dfrac{t^2+4}{t-2}$

37. $\lim_{t\to 2} \dfrac{t^2-4}{t-2}$

38. $\lim_{x\to 2^-} \dfrac{x}{x-2}$

39. $\lim_{x\to 1^+} \dfrac{x}{x-1}$

40. $\lim_{x\to 0} \dfrac{|x|}{|x|+1}$

41. $\lim_{x\to -1^-} \dfrac{1}{x+1}$

42. $\lim_{x\to 0} \dfrac{1}{|x|}$

43. $\lim_{x\to -2^+} \dfrac{1}{x+2}$

44. $\lim_{x\to 3^-} \dfrac{x^2}{x-3}$

45. $\lim_{x\to 3} \dfrac{x-3}{x^2}$

46. $\lim_{x\to 1^+} \dfrac{2}{x^2-1}$

47. $\lim_{x\to 2^-} \dfrac{x^2+5}{x-2}$

48. $\lim_{x\to 2} \dfrac{x-2}{x^2+5}$

49. $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$

50. $\lim_{x\to 1^+} \dfrac{x+4}{x^2+2x-3}$

51. Find

$$\lim \frac{x-1}{2x^2-7x+5}$$

as (a) $x \to 0$, (b) $x \to \infty$, and (c) $x \to 1$.

52. Find the domain and range of the function

$$y = \sqrt{\frac{1}{x} - 1}.$$

Sketch the graphs of the functions in Problems 53–56.

53. $f(x) = \dfrac{1}{x-2}$

54. $f(x) = \dfrac{1}{x+1}$

55. $f(x) = 1 + \dfrac{1}{x}$

56. $f(x) = \dfrac{1}{|x|}$

57. Find $\lim_{x\to\infty} f(x)$ if

$$\frac{2x-3}{x} < f(x) < \frac{2x^2+5x}{x^2}.$$

58. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ a polynomial of degree m. Show that $\lim_{x\to\infty} f(x)/g(x)$ is a_n/b_m if $m = n$, 0 if $m > n$, infinite if $m < n$. (*Hint:* Divide the numerator and denominator of the fraction by x^m. What happens to x^n/x^m as $x \to \infty$ if $m = n$? If $m > n$? If $m < n$?)

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1.10 INFINITY AS A LIMIT

1. $$\lim_{x\to\infty} \frac{2x+3}{5x+7} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}$$

3. $$\lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} = 0$$

5. $$\lim_{y\to\infty} \frac{3y+7}{y^2-2} = \lim_{y\to\infty} \frac{\frac{3}{y}+\frac{7}{y^2}}{1-\frac{2}{y^2}} = 0.$$

7. $$\lim_{t\to\infty} \frac{t^2-2t+3}{2t^2+5t-3} = \lim_{t\to\infty} \frac{1-\frac{2}{t}+\frac{3}{t^2}}{2+\frac{5}{t}-\frac{3}{t^2}} = \frac{1}{2}$$

9. $\lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}} = 1$

11. $\lim_{x\to-\infty} |x| = \infty$

13. $\lim_{a\to\infty} \frac{|a|}{|a|+1} = \lim_{a\to\infty} \frac{1}{1+\frac{1}{|a|}} = 1$

15. $\lim_{x\to\infty} \frac{3x^3+5x^2-7}{10x^3-11x+5} = \lim_{x\to\infty} \frac{3+\frac{5}{x}-\frac{7}{x^3}}{10-\frac{11}{x^2}+\frac{5}{x^3}} = \frac{3}{10}$

17. $\lim_{s\to\infty} \left(\frac{s}{s+1}\right)\left(\frac{s^2}{5+s^2}\right) = \lim_{s\to\infty} \left(\frac{1}{1+\frac{1}{s}}\right)\left(\frac{1}{\frac{5}{s^2}+1}\right) = 1$

19. $\lim_{r\to-\infty} \frac{8r^2+7r}{4r^2} = \lim_{r\to-\infty} \frac{8+\frac{7}{r}}{4} = 2$

21. $\lim_{y\to\infty} \frac{y^4}{y^4-7y^3+7y^2+9} = \lim_{y\to\infty} \frac{1}{1-\frac{7}{y}+\frac{7}{y^2}+\frac{9}{y^4}} = 1$

23. $\lim_{x\to\infty} \frac{x-3}{x^2-5x+4} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{3}{x^2}}{1-\frac{5}{x}+\frac{4}{x^2}} = 0$

25. $\lim_{x\to\infty} \frac{-2x^3-2x+3}{3x^3+3x^2-5x} = \lim_{x\to\infty} \frac{-2-\frac{2}{x^2}+\frac{3}{x^3}}{3+\frac{3}{x}-\frac{5}{x^2}} = -\frac{2}{3}$

27. $\lim_{x\to\infty} \frac{x+\sin x}{x+\cos x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{1+\frac{\cos x}{x}} = 1$

29. $\lim_{x\to\infty} \left(\frac{1}{x^4}+\frac{1}{x}\right) = 0$

31. $\lim_{x\to\infty} \left(\cos\frac{1}{x}+1\right) = \lim_{y\to 0} (\cos y+1) = 2$

33. $\lim_{x\to 0^+} \frac{1}{3x} = \infty$

35. $\lim_{x\to 0^+} \frac{5}{2x} = \infty$

37. $\lim_{t\to 2} \frac{t^2-4}{t-2} = \lim_{t\to 2} (t+2) = 4$

39. $\lim_{x\to 1^+} \frac{x}{x-1} = \infty$

41. $\lim_{x\to 1^-} \frac{1}{x+1} = -\infty$

43. $\lim_{x\to -2^+} \frac{1}{x+2} = \infty$

45. $\lim_{x\to 3} \frac{x-3}{x^2} = \frac{0}{9} = 0$

47. $\lim_{x\to 2^-} \frac{x^2+5}{x-2} = -\infty$

49. $\lim_{x\to -5} \frac{x^2+3x-10}{x+5} = \lim_{x\to -5} \frac{(x+5)(x-2)}{x+5} = -7$

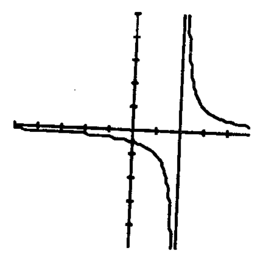
51. (a) $\lim_{x\to 0} \frac{x-1}{2x^2-7x+5} = -\frac{1}{5}$

(b) $\lim_{x\to \infty} \frac{x-1}{2x^2-7x+5} = 0$

(c) $\lim_{x\to 1} \frac{x-1}{2x^2-7x+5} = -\frac{1}{3}$

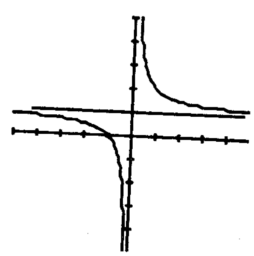
53. $f(x) = \frac{1}{x-2}$ has a vertical asymptote at $x = 2$ because $\lim_{x\to 2^+} f(x) = +\infty$ and $\lim_{x\to 2^-} f(x) = -\infty$.

f has a horizontal asymptote at $y = 0$ because $\lim_{x\to \infty} f(x) = 0$.



55. $f(x) = 1 + \frac{1}{x}$ has a vertical asymptote at $x = 0$ because $\lim_{x\to 0^+} f(x) = +\infty$ and $\lim_{x\to 0^-} f(x) = -\infty$.

f has a horizontal asymptote at $y = 1$ because $\lim_{x\to \infty} f(x) = 1$.



57. $\frac{2x-3}{x} < f(x) < \frac{2x^2+5x}{x^2} \Rightarrow \lim_{x\to\infty} \frac{2x-3}{x} < \lim_{x\to\infty} f(x) < \lim_{x\to\infty} \frac{2x^2+5x}{x^2}$.

Since $\lim_{x\to\infty} \frac{2x-3}{x} = 2$ and $\lim_{x\to\infty} \frac{2x^2+5x}{x^2} = 2$,

by the Squeeze Theorem, $\lim_{x\to\infty} f(x) = 2$.