

$$\int u \cdot dv = uv - \int v \cdot du$$

For u : Choose something that becomes simpler when differentiated.
 For v : Choose something whose integral is simple.

Or try using tabular integration (see example below):

$x \, dx$).
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The products of the functions connected by the arrows are added, with the middle sign changed, to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

Notice the agreement with Example 4.

EXAMPLE 7 Evaluate

$$\int x^3 \sin x \, dx$$

by tabular integration.

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we have

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	sin x
$3x^2$	(-)	$-\cos x$
$6x$	(+)	$-\sin x$
6	(-)	$\cos x$
0		sin x .

The products of the functions connected by the arrows are added, with every other sign changed, to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

PROBLEMS

In Problems 1–38, evaluate the integrals.

- | | | | |
|------------------------------|-------------------------------|--|---|
| 1. $\int x \sin x \, dx$ | 2. $\int x \cos 2x \, dx$ | 15. $\int x^3 e^x \, dx$ | 16. $\int x^4 e^{-x} \, dx$ |
| 3. $\int x^2 \sin x \, dx$ | 4. $\int x^2 \cos x \, dx$ | 17. $\int (x^2 - 5x)e^x \, dx$ | 18. $\int (x^2 + x + 1)e^x \, dx$ |
| 5. $\int_1^2 x \ln x \, dx$ | 6. $\int x^2 \ln x \, dx$ | 19. $\int x^5 e^x \, dx$ | 20. $\int x^2 e^{4x} \, dx$ |
| 7. $\int x^3 \ln x \, dx$ | 8. $\int_0^1 \ln(x+1) \, dx$ | 21. $\int_0^{\pi/2} x^2 \sin 2x \, dx$ | 22. $\int_0^{\pi/2} x^3 \cos 2x \, dx$ |
| 9. $\int \tan^{-1} x \, dx$ | 10. $\int \tan^{-1} ax \, dx$ | 23. $\int x^4 \cos x \, dx$ | 24. $\int x^5 \sin x \, dx$ |
| 11. $\int \sin^{-1} x \, dx$ | 12. $\int \sin^{-1} ax \, dx$ | 25. $\int x^2 \cos ax \, dx$ | 26. $\int x \cos(2x+1) \, dx$ |
| 13. $\int x \sec^2 x \, dx$ | 14. $\int 4x \sec^2 2x \, dx$ | 27. $\int_1^2 x \sec^{-1} x \, dx$ | 28. $\int_1^4 \sec^{-1} \sqrt{x} \, dx$ |

7.2 INTEGRATION BY PARTS

1. Let
- $u = x$
- and
- $dv = \sin x \, dx \Rightarrow du = dx$
- and
- $v = -\cos x$

$$\int x \sin x \, dx = -x \cos x - \int (-\cos x) \, dx = -x \cos x + \sin x + C$$

- 3.
- $\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$

$$\begin{array}{r} x^2 \quad + \quad \sin x \\ 2x \quad - \quad -\cos x \\ 2 \quad + \quad -\sin x \\ 0 \quad - \quad \cos x \end{array}$$

5. Let
- $u = \ln x$
- and
- $dv = x \, dx \Rightarrow du = \frac{1}{x} \, dx$
- and
- $v = \frac{1}{2}x^2$

$$\int_1^2 \ln x \, dx = \left[\frac{1}{2}x^2 \ln x \right]_1^2 - \int_1^2 \frac{1}{2}x \, dx = \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 \right]_1^2 = 2 \ln 2 - \frac{3}{4}$$

7. Let
- $u = \ln x$
- and
- $dv = x^3 \, dx \Rightarrow du = \frac{1}{x} \, dx$
- and
- $v = \frac{1}{4}x^4$

$$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

9. Let
- $u = \tan^{-1} x$
- and
- $dv = dx \Rightarrow du = \frac{1}{1+x^2} \, dx$
- and
- $v = x$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x \, dx}{1+x^2} = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

11. Let
- $u = \sin^{-1} x$
- and
- $dv = dx \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$
- and
- $v = x$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x \, dx}{\sqrt{1-x^2}} = x \sin^{-1} x + \sqrt{1-x^2} + C$$

13. Let
- $u = x$
- and
- $dv = \sec^2 x \, dx \Rightarrow du = dx$
- and
- $v = \tan x$

$$\int x \sec^2 x \, dx = x \tan x - \int \tan x \, dx = x \tan x + \ln |\cos x| + C$$

- 15.
- $\int x^3 e^x \, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = e^x (x^3 - 3x^2 + 6x - 6) + C$

$$\begin{array}{r} x^3 \quad + \quad e^x \\ 3x^2 \quad - \quad e^x \\ 6x \quad + \quad e^x \\ 6 \quad - \quad e^x \\ 0 \quad - \quad e^x \end{array}$$

$$17. \int (x^2 - 5x)e^x dx = (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x - 2e^x + C$$

$$\begin{array}{r} x^2 - 5x \\ 2x - 5 \\ 2 \\ 0 \end{array} \begin{array}{l} + \\ - \\ + \\ - \end{array} \begin{array}{l} e^x \\ e^x \\ e^x \\ e^x \end{array} = e^x(x^2 - 7x + 7) + C$$

$$19. \int x^5 e^x dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120e^x + C$$

$$e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$$

$$\begin{array}{r} x^5 \\ 5x^4 \\ 20x^3 \\ 60x^2 \\ 120x \\ 120 \\ 0 \end{array} \begin{array}{l} + \\ - \\ + \\ - \\ + \\ - \\ + \end{array} \begin{array}{l} e^x \\ e^x \\ e^x \\ e^x \\ e^x \\ e^x \\ e^x \end{array}$$

$$21. \int_0^{\frac{\pi}{2}} x^2 \sin 2x dx = -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8} - \frac{1}{2}$$

$$\begin{array}{r} x^2 \\ 2x \\ 2 \\ 0 \end{array} \begin{array}{l} + \\ - \\ + \\ - \end{array} \begin{array}{l} \sin 2x \\ -\frac{1}{2} \cos 2x \\ -\frac{1}{4} \sin 2x \\ \frac{1}{8} \cos 2x \end{array}$$

$$23. \int x^4 \cos x dx = x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x + C$$

$$\begin{array}{r} x^4 \\ 4x^3 \\ 12x^2 \\ 24x \\ 24 \\ 0 \end{array} \begin{array}{l} + \\ - \\ + \\ - \\ + \\ - \end{array} \begin{array}{l} \cos x \\ \sin x \\ -\cos x \\ -\sin x \\ \cos x \\ \sin x \end{array}$$

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$$\int x^2 \cos ax \, dx = \frac{x^2}{a} \sin ax + \frac{2x}{a^2} \cos ax - \frac{2}{a^3} \sin ax + C$$

$$\begin{array}{rcl} x^2 & + & \cos ax \\ & \searrow & \rightarrow \frac{1}{a} \sin ax \\ 2x & - & \\ & \searrow & \rightarrow -\frac{1}{a^2} \cos ax \\ 2 & + & \\ & \searrow & \rightarrow -\frac{1}{a^3} \sin ax \\ 0 & & \end{array}$$

27. Let $u = \sec^{-1} x$ and $dv = x \, dx \Rightarrow du = \frac{dx}{x\sqrt{x^2-1}}$ and $v = \frac{1}{2}x^2$

$$\int_1^2 x \sec^{-1} x \, dx = \left[\frac{1}{2}x^2 \sec^{-1} x \right]_1^2 - \frac{1}{2} \int_1^2 \frac{x}{\sqrt{x^2-1}} \, dx =$$

$$\left[\frac{1}{2}x^2 \sec^{-1} x - \frac{1}{2}\sqrt{x^2-1} \right]_1^2 = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

29. $\int_1^e \frac{\ln x}{x} \, dx = \left[\frac{1}{2} \ln^2 x \right]_1^e = \frac{1}{2}$

31. Let $u = \sin^{-1}\left(\frac{1}{x}\right)$ and $dv = x \, dx \Rightarrow du = \frac{-dx}{x\sqrt{x^2-1}}$ and $v = \frac{1}{2}x^2$

$$\int x \sin^{-1}\left(\frac{1}{x}\right) \, dx = \frac{1}{2}x^2 \sin^{-1}\left(\frac{1}{x}\right) + \int \frac{x \, dx}{\sqrt{x^2-1}} = \frac{1}{2}x^2 \sin^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\sqrt{x^2-1} + C$$

33. Let $u = e^x$ and $dv = \sin x \, dx \Rightarrow du = e^x \, dx$ and $v = -\cos x$.

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Now let $u = e^x$ and $dv = \cos x \, dx \Rightarrow du = e^x \, dx$ and $v = \sin x$.

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

35. Let $u = e^{2x}$ and $dv = \cos 3x \, dx \Rightarrow du = 2e^{2x} \, dx$ and $v = \frac{1}{3} \sin 3x$

$$\int e^{2x} \cos 3x \, dx = \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x \, dx$$

Now let $u = e^{2x}$ and $dv = \sin 3x \, dx \Rightarrow du = 2e^{2x} \, dx$ and $v = -\frac{1}{3} \cos 3x$.

$$\int e^{2x} \cos 3x \, dx = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cos 3x \, dx$$

$$\frac{13}{9} \int e^{2x} \cos 3x \, dx = \frac{1}{9} e^{2x} (3 \sin 3x + 2 \cos 3x)$$

$$\int e^{2x} \cos 3x \, dx = \frac{1}{13} e^{2x} (3 \sin 3x + 2 \cos 3x) + C$$

37. Let $u = \sin(\ln x)$ and $dv = dx \Rightarrow du = \cos(\ln x) \frac{1}{x} dx$ and $v = x$

$$\int \sin(\ln x) dx = x \sin(\ln x) - \int \cos(\ln x) dx$$

Now let $u = \cos(\ln x)$ and $dv = dx \Rightarrow du = -\sin(\ln x) \frac{1}{x} dx$ and $v = x$

$$= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx$$

$$2 \int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x)$$

$$\int \sin(\ln x) dx = \frac{x}{2} (\sin(\ln x) - \cos(\ln x)) + C$$

39. (a) $\int_0^{\pi} x \sin x \, dx = -x \cos x + \sin x \Big|_0^{\pi} = \pi$ (See problem 1)

(b) $\int_{\pi}^{2\pi} |x \sin x| \, dx = \int_{\pi}^{2\pi} (-x \sin x) \, dx = x \cos x - \sin x \Big|_{\pi}^{2\pi} = 3\pi$

41. $v = 2\pi \int_0^1 x e^{-x} \, dx = 2\pi (-x e^{-x} - e^{-x}) \Big|_0^1 = 2\pi \left(\frac{e-2}{2} \right)$

43. $A = \int_0^1 x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x \Big|_0^1 = e - 2$

$$\bar{x} = \frac{1}{e-2} \int_0^1 x^3 e^x \, dx = \frac{1}{e-2} [e^x (x^3 - 3x^2 + 6x - 6)] \Big|_0^1 = \frac{6-2e}{2-e}$$

$$\bar{y} = \frac{1}{e-2} \int_0^1 \frac{1}{2} x^4 e^{2x} \, dx = \frac{1}{2(e-2)} \left[e^{2x} \left(\frac{x^4}{2} - x^3 + \frac{3x^2}{2} - \frac{3x}{2} + \frac{3}{4} \right) \right] \Big|_0^1$$

$$= \frac{e^2 - 3}{8(e-2)}$$

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$$45. (a) V = \pi \int_0^{\pi} x^2 \sin^2 x \, dx = \left[\frac{x^3}{2} - \frac{x^2 \sin 2x}{4} \right]_0^{\pi} - \int_0^{\pi} \left(x^2 - \frac{x \sin 2x}{2} \right) dx$$

$$\text{Let } u = x^2 \quad dv = \sin^2 x \, dx \quad \text{Let } u = \frac{x}{2} \quad dv = \sin 2x \, dx$$

$$du = 2x \, dx \quad v = \frac{x}{2} - \frac{\sin 2x}{4} \quad du = \frac{1}{2} dx \quad v = -\frac{1}{2} \cos 2x \, dx$$

$$= \left[\frac{x^3}{2} - \frac{x^2 \sin 2x}{4} - \frac{x^3}{3} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8} \right]_0^{\pi} = \frac{\pi^4}{6} - \frac{\pi^2}{4}$$

$$(b) V = 2\pi \int_0^{\pi} (\pi - x) x \sin x \, dx$$

$$= 2\pi^2 [-x \cos x + \sin x]_0^{\pi} - 2\pi [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^{\pi} = 8\pi$$

$$47. M_y = x \delta \, dA = \int_0^{\pi} x(1+x) \sin x \, dx$$

$$= -(x+x^2) \cos x + (1+2x) \sin x + 2 \cos x \Big|_0^{\pi} = \pi^2 + \pi - 4$$

$$49. x = e^t \sin t \Rightarrow \frac{dx}{dt} = e^t (\sin t + \cos t).$$

$$y = e^t \cos t \Rightarrow \frac{dy}{dt} = e^t (-\sin t + \cos t). \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= e^t \sqrt{2} \, dt. \quad S = \int 2\pi r \, ds = 2\pi \sqrt{2} \int_0^{\pi} e^{2t} \cos t \, dt$$

$$= \frac{2\pi \sqrt{2}}{5} e^{2t} (\sin t + 2 \cos t) \Big|_0^{\pi} = \frac{2\pi \sqrt{2} (e^{\pi} - 2)}{5}$$

$$51. \text{ Let } u = \tan^{-1} x \Rightarrow du = \frac{dx}{1+x^2} \text{ and } dv = x \, dx \Rightarrow v = \frac{x^2}{2} + C. \text{ Take } C = \frac{1}{2}.$$

$$\int x \tan^{-1} x \, dx = \frac{x^2+1}{2} \tan^{-1} x - \int \frac{x^2+1}{2} \cdot \frac{dx}{1+x^2} = \frac{x^2+1}{2} \tan^{-1} x - \frac{x}{2} + C$$

7.3 PRODUCTS AND POWERS OF TRIGONOMETRIC FUNCTIONS

$$1. \int \sin^5 x \, dx = \int (\sin^2 x)^2 \sin x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C$$

$$3. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 x \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 x) \cos x \, dx = \left[\sin x - \frac{1}{3} \sin^3 x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4}{3}$$