$$\int u \cdot dv = uv - \int v \cdot du$$

For u: Choose something that becomes simpler when differentiated.
For v: Choose something whose integral is simple.

Or try using tabular integration (see example below):

The products of the functions connected by the arrows are added, with the middle sign changed, to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Notice the agreement with Example 4. ∎

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EXAMPLE Evaluate

$$\int x^3 \sin x \, dx$$

by tabular integration.

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we have

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\sin x$
$3x^2$	(−)	$-\cos x$
$6x$	(+)	$-\sin x$
6	(−)	$\cos x$
0		$\sin x$.

The products of the functions connected by the arrows are added, with every other sign changed, to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \quad ∎$$

PROBLEMS

In Problems 1–38, evaluate the integrals.

1. $\int x \sin x \, dx$
2. $\int x \cos 2x \, dx$
3. $\int x^2 \sin x \, dx$
4. $\int x^2 \cos x \, dx$
5. $\int_1^2 x \ln x \, dx$
6. $\int x^2 \ln x \, dx$
7. $\int x^3 \ln x \, dx$
8. $\int_0^1 \ln(x + 1) \, dx$
9. $\int \tan^{-1} x \, dx$
10. $\int \tan^{-1} ax \, dx$
11. $\int \sin^{-1} x \, dx$
12. $\int \sin^{-1} ax \, dx$
13. $\int x \sec^2 x \, dx$
14. $\int 4x \sec^2 2x \, dx$
15. $\int x^3 e^x \, dx$
16. $\int x^4 e^{-x} \, dx$
17. $\int (x^2 - 5x) e^x \, dx$
18. $\int (x^2 + x + 1) e^x \, dx$
19. $\int x^5 e^x \, dx$
20. $\int x^2 e^{4x} \, dx$
21. $\int_0^{\pi/2} x^2 \sin 2x \, dx$
22. $\int_0^{\pi/2} x^3 \cos 2x \, dx$
23. $\int x^4 \cos x \, dx$
24. $\int x^5 \sin x \, dx$
25. $\int x^2 \cos ax \, dx$
26. $\int x \cos(2x + 1) \, dx$
27. $\int_1^2 x \sec^{-1} x \, dx$
28. $\int_1^4 \sec^{-1} \sqrt{x} \, dx$

7.2 INTEGRATION BY PARTS

1. Let $u = x$ and $dv = \sin x\,dx \Rightarrow du = dx$ and $v = -\cos x$

$$\int x \sin x\,dx = -x\cos x - \int(-\cos x)\,dx = -x\cos x + \sin x + C$$

3. $\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

x^2	+	$\sin x$
$2x$	−	$-\cos x$
2	+	$-\sin x$
0		$\cos x$

5. Let $u = \ln x$ and $dv = x\,dx \Rightarrow du = \frac{1}{x}\,dx$ and $v = \frac{1}{2}x^2$

$$\int_1^2 \ln x\,dx = \frac{1}{2}x^2\ln x\Big]_1^2 - \int_1^2 \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2\Big]_1^2 = 2\ln 2 - \frac{3}{4}.$$

7. Let $u = \ln x$ and $dv = x^3\,dx \Rightarrow du = \frac{1}{x}\,dx$ and $v = \frac{1}{4}x^4$

$$\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$$

9. Let $u = \tan^{-1}x$ and $dv = dx \Rightarrow du = \frac{1}{1+x^2}\,dx$ and $v = x$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x\,dx}{1+x^2} = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

11. Let $u = \sin^{-1}x$ and $dv = dx \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$ and $v = x$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int\frac{x\,dx}{\sqrt{1-x^2}} = x\sin^{-1}x + \sqrt{1-x^2} + C$$

13. Let $u = x$ and $dv = \sec^2 x\,dx \Rightarrow du = dx$ and $v = \tan x$

$$\int x\sec^2 x\,dx = x\tan x - \int\tan x\,dx = x\tan x + \ln|\cos x| + C$$

15. $\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C = e^x(x^3 - 3x^2 + 6x - 6) + C$

x^3	+	e^x
$3x^2$	−	e^x
$6x$	+	e^x
6	−	e^x
0		e^x

17. $\int (x^2 - 5x)e^x dx = (x^2 - 5x)e^x - (2x - 5)e^x + 2e^x - 2e^x + C$

$= e^x(x^2 - 7x + 7) + C$

$x^2 - 5x$	+	e^x
$2x - 5$	−	e^x
2	+	e^x
0		e^x

19. $\int x^5 e^x dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + C$

$e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120) + C$

x^5	+	e^x
$5x^4$	−	e^x
$20x^3$	+	e^x
$60x^2$	−	e^x
$120x$	+	e^x
120	−	e^x
0		e^x

21. $\int_0^{\frac{\pi}{2}} x^2 \sin 2x\, dx = -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x \Big]_0^{\frac{\pi}{2}} = \frac{\pi^2}{8} - \frac{1}{2}$

x^2	+	$\sin 2x$
$2x$	−	$-\frac{1}{2}\cos 2x$
2	+	$-\frac{1}{4}\sin 2x$
0		$\frac{1}{8}\cos 2x$

23. $\int x^4 \cos x\, dx = x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x + C$

x^4	+	$\cos x$
$4x^3$	−	$\sin x$
$12x^2$	+	$-\cos x$
$24x$	−	$-\sin x$
24	+	$\cos x$
0		$\sin x$

$$\int x^2 \cos ax\,dx = \frac{x^2}{a}\sin ax + \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + C$$

x^2	$+$	$\cos ax$
$2x$	$-$	$\frac{1}{a}\sin ax$
2	$+$	$-\frac{1}{a^2}\cos ax$
0		$-\frac{1}{a^3}\sin ax$

27. Let $u = \sec^{-1} x$ and $dv = x\,dx \Rightarrow du = \dfrac{dx}{x\sqrt{x^2-1}}$ and $v = \frac{1}{2}x^2$

$$\int_1^2 x \sec^{-1} x\,dx = \frac{1}{2}x^2 \sec^{-1} x\Big]_1^2 - \frac{1}{2}\int_1^2 \frac{x}{\sqrt{x^2-1}}dx =$$

$$\frac{1}{2}x^2 \sec^{-1} x - \frac{1}{2}\sqrt{x^2-1}\Big]_1^2 = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

29. $\displaystyle\int_1^e \frac{\ln x}{x}dx = \frac{1}{2}\ln^2 x\Big]_1^e = \frac{1}{2}$

31. Let $u = \sin^{-1}\left(\frac{1}{x}\right)$ and $dv = x\,dx \Rightarrow du = \dfrac{-dx}{x\sqrt{x^2-1}}$ and $v = \frac{1}{2}x^2$

$$\int x \sin^{-1}\left(\frac{1}{x}\right)dx = \frac{1}{2}x^2 \sin^{-1}\left(\frac{1}{x}\right) + \int \frac{x\,dx}{\sqrt{x^2-1}} = \frac{1}{2}x^2 \sin^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\sqrt{x^2-1} + C$$

33. Let $u = e^x$ and $dv = \sin x\,dx \Rightarrow du = e^x\,dx$ and $v = -\cos x$.

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$$

Now let $u = e^x$ and $dv = \cos x\,dx \Rightarrow du = e^x\,dx$ and $v = \sin x$.

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x)$$

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

35. Let $u = e^{2x}$ and $dv = \cos 3x\,dx \Rightarrow du = 2e^{2x}\,dx$ and $v = \frac{1}{3}\sin 3x$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x\,dx$$

Now let $u = e^{2x}$ and $dv = \sin 3x\,dx \Rightarrow du = 2e^{2x}\,dx$ and $v = -\frac{1}{3}\cos 3x$.

$$\int e^{2x}\cos 3x\,dx = \frac{1}{3}e^{2x}\sin 3x + \frac{2}{9}e^{2x}\cos 3x - \frac{4}{9}\int e^{2x}\cos 3x\,dx$$

$$\frac{13}{9}\int e^{2x}\cos 3x\,dx = \frac{1}{9}e^{2x}(3\sin 3x + 2\cos 3x)$$

$$\int e^{2x}\cos 3x\,dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C$$

37. Let $u = \sin(\ln x)$ and $dv = dx \Rightarrow du = \cos(\ln x)\frac{1}{x}dx$ and $v = x$

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)$$

Now let $u = \cos(\ln x)$ and $dv = dx \Rightarrow du = -\sin(\ln x)\frac{1}{x}dx$ and $v = x$

$$\pm x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

$$2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x)dx = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$$

39. (a) $\int_0^{\pi} x\sin x\,dx = -x\cos x + \sin x\Big]_0^{\pi} = \pi$ (See problem 1)

(b) $\int_{\pi}^{2\pi} |x\sin x|\,dx = \int_{\pi}^{2\pi}(-x\sin x)\,dx = x\cos x - \sin x\Big]_{\pi}^{2\pi} = 3\pi$

41. $V = 2\pi\int_0^1 xe^{-x}\,dx = 2\pi(-xe^{-x} - e^{-x})\Big]_0^1 = 2\pi\left(\frac{e-2}{2}\right)$

43. $A = \int_0^1 x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x\Big]_0^1 = e - 2$

$$\bar{x} = \frac{1}{e-2}\int_0^1 x^3e^x\,dx = \frac{1}{e-2}[e^x(x^3 - 3x^2\ \ 6x - 6)\Big]_0^1 = \frac{6-2e}{2-e}$$

$$\bar{y} = \frac{1}{e-2}\int_0^1 \frac{1}{2}x^4e^{2x}\,dx = \frac{1}{2(e-2)}\left[e^{2x}\left(\frac{x^4}{2} - x^3 + \frac{3x^2}{2} - \frac{3x}{2} + \frac{3}{4}\right)\right]_0^1$$

$$= \frac{e^2-3}{8(e-2)}$$

45. (a) $V = \pi\int_0^{\pi} x^2 \sin^2 x\,dx = \frac{x^3}{2} - \frac{x^2 \sin 2x}{4}\Big]_0^{\pi} - \int_0^{\pi}\left(x^2 - \frac{x \sin 2x}{2}\right)dx$

Let $u = x^2 \quad dv = \sin^2 x\,dx \qquad$ Let $u = \frac{x}{2} \quad dv = \sin 2x\,dx$

$du = 2x\,dx \quad v = \frac{x}{2} - \frac{\sin 2x}{4} \qquad du = \frac{1}{2}dx \quad v = -\frac{1}{2}\cos 2x\,dx$

$$= \frac{x^3}{2} - \frac{x^2 \sin 2x}{4} - \frac{x^3}{3} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8}\Big]_0^{\pi} = \frac{\pi^4}{6} - \frac{\pi^2}{4}$$

(b) $V = 2\pi\int_0^{\pi} (\pi - x)\, x \sin x\,dx$

$$= 2\pi^2\,[-x \cos x + \sin x\Big]_0^{\pi} - 2\pi\,[-x^2 \cos x + 2x \sin x + 2 \cos x\Big]_0^{\pi} = 8\pi$$

47. $M_y = x\,\delta\,dA = \int_0^{\pi} x(1 + x)\sin x\,dx$

$$= -(x + x^2)\cos x + (1 + 2x)\sin x + 2\cos x\Big]_0^{\pi} = \pi^2 + \pi - 4$$

49. $x = e^t \sin t \Rightarrow \frac{dx}{dt} = e^t(\sin t + \cos t)$.

$y = e^t \cos t \Rightarrow \frac{dy}{dt} = e^t(-\sin t + \cos t)$. $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= e^t\sqrt{2}\,dt$. $S = \int 2\pi r\,ds \; = 2\pi\sqrt{2}\int_0^{\pi} e^{2t}\cos t\,dt$

$$= \frac{2\pi\sqrt{2}}{5}e^{2t}(\sin t + 2\cos t)\Big]_0^{\pi} = \frac{2\pi\sqrt{2}\,(e^{\pi} - 2)}{5}$$

51. Let $u = \tan^{-1}x \Rightarrow du = \frac{dx}{1 + x^2}$ and $dv = x\,dx \Rightarrow v = \frac{x^2}{2} + C$. Take $C = \frac{1}{2}$.

$$\int x\tan^{-1}x\,dx = \frac{x^2 + 1}{2}\tan^{-1}x - \int \frac{x^2 + 1}{2}\cdot\frac{dx}{1 + x^2} = \frac{x^2 + 1}{2}\tan^{-1}x - \frac{x}{2} + C$$

7.3 PRODUCTS AND POWERS OF TRIGONOMETRIC FUNCTIONS

1. $\int \sin^5 x\,dx = \int (\sin^2 x)^2 \sin x\,dx = \int (1 - \cos^2 x)^2 \sin x\,dx$

$$= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

3. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 x\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 x)\cos x\,dx = \sin x - \frac{1}{3}\sin^3 x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4}{3}$