

TABLE 3.1

Derivatives

$$1. \frac{d(\sin^{-1}u)}{dx} = \frac{du/dx}{\sqrt{1-u^2}}, \quad -1 < u < 1$$

$$2. \frac{d(\cos^{-1}u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}, \quad -1 < u < 1$$

$$3. \frac{d(\tan^{-1}u)}{dx} = \frac{du/dx}{1+u^2}$$

$$4. \frac{d(\cot^{-1}u)}{dx} = -\frac{du/dx}{1+u^2}$$

$$5. \frac{d(\sec^{-1}u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

$$6. \frac{d(\csc^{-1}u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

Differentials

$$1'. d(\sin^{-1}u) = \frac{du}{\sqrt{1-u^2}}, \quad -1 < u < 1$$

$$2'. d(\cos^{-1}u) = -\frac{du}{\sqrt{1-u^2}}, \quad -1 < u < 1$$

$$3'. d(\tan^{-1}u) = \frac{du}{1+u^2}$$

$$4'. d(\cot^{-1}u) = -\frac{du}{1+u^2}$$

$$5'. d(\sec^{-1}u) = \frac{du}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

$$6'. d(\csc^{-1}u) = \frac{-du}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

PROBLEMS

Problems 1–20, find dy/dx .

1. $y = \cos^{-1}x^2$
2. $y = \cos^{-1}(1/x)$
3. $y = 5 \tan^{-1}3x$
4. $y = \cot^{-1}\sqrt{x}$
5. $y = \sin^{-1}(x/2)$
6. $y = \sin^{-1}(1-x)$
7. $y = \sec^{-1}5x$
8. $y = (1/3) \tan^{-1}(x/3)$
9. $y = \csc^{-1}(x^2 + 1)$
10. $y = \cos^{-1}2x$
11. $y = \csc^{-1}\sqrt{x} + \sec^{-1}\sqrt{x}$
12. $y = \csc^{-1}\sqrt{x+1}$
13. $y = \cot^{-1}\sqrt{x-1}$
14. $y = x\sqrt{1-x^2} - \cos^{-1}x$
15. $y = \sqrt{x^2-4} - 2 \sec^{-1}(x/2)$
16. $y = \cot^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$
17. $y = \tan^{-1}\frac{x-1}{x+1}$
18. $y = x \sin^{-1}x + \sqrt{1-x^2}$
19. $y = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1-x^2} \sin^{-1}x$
20. $y = x \cos^{-1}2x - (1/2)\sqrt{1-4x^2}$

Evaluate the integrals in Problems 21–40.

21. $\int_{-1}^{1/2} \frac{dx}{\sqrt{1-x^2}}$
22. $\int_{-1}^1 \frac{dx}{1+x^2}$
23. $\int_{-2}^2 \frac{dx}{x\sqrt{x^2-1}}$
24. $\int_{-2}^{-\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}}$
25. $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{6 dx}{1+x^2}$

$$27. \int_0^{\sqrt{2}/2} \frac{x dx}{\sqrt{1-x^4}}$$

$$29. \int_{1/\sqrt{3}}^1 \frac{dx}{x\sqrt{4x^2-1}}$$

$$31. \int_0^{\sqrt{2}} \frac{4x dx}{\sqrt{4-x^4}}$$

$$33. \int_{\sqrt{2}}^{\sqrt[3]{2}} \frac{x dx}{x^2\sqrt{x^4-1}}$$

$$34. \int_2^4 \frac{dx}{2x\sqrt{x-1}} \quad (\text{Hint: Let } x = u^2.)$$

$$35. \int_0^2 \frac{dx}{1+(x-1)^2}$$

$$37. \int_{1/2}^{3/4} \frac{dx}{\sqrt{x}\sqrt{1-x}}$$

$$39. \int_{-2/3}^{-\sqrt{2}/3} \frac{dx}{x\sqrt{9x^2-1}}$$

$$28. \int_0^{1/4} \frac{dx}{\sqrt{1-4x^2}}$$

$$30. \int_0^1 \frac{x dx}{1+x^4}$$

$$32. \int_0^1 \frac{dx}{\sqrt{4-x^2}}$$

$$36. \int_1^3 \frac{2 dx}{\sqrt{x}(1+x)}$$

$$38. \int_{3/2}^{1+\sqrt{2}/2} \frac{dx}{\sqrt{1-(x-1)^2}}$$

$$40. \int_{-\pi/2}^{\pi/2} \frac{2 \cos x dx}{1+\sin^2x}$$

Use l'Hôpital's rule (Article 3.8) to evaluate the limits in Problems 41–44.

$$41. \lim_{x \rightarrow 0} \frac{\sin^{-1}2x}{x}$$

$$42. \lim_{x \rightarrow 0} \frac{2 \tan^{-1}3x}{5x}$$

6.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$1. \quad y = \cos^{-1} x^2 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2)^2}} \frac{d}{dx}(x^2) = \frac{-2x}{\sqrt{1-x^4}}.$$

$$3. \quad y = 5 \tan^{-1} 3x \Rightarrow \frac{dy}{dx} = 5 \left(\frac{1}{1+(3x)^2} \right) \frac{d}{dx}(3x) = \frac{15}{1+9x^2}.$$

$$5. \quad y = \sin^{-1} \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{\sqrt{4-x^2}}.$$

$$7. \quad y = \sec^{-1} 5x \Rightarrow \frac{dy}{dx} = \frac{1}{|5x|\sqrt{(5x)^2-1}} \frac{d}{dx}(5x) = \frac{1}{|x|\sqrt{25x^2-1}}.$$

$$9. \quad y = \csc^{-1}(x^2 + 1) \Rightarrow \frac{dy}{dx} = \frac{-1}{|x^2 + 1| \sqrt{(x^2 + 1)^2 - 1}} \frac{d}{dx}(x^2 + 1) = \frac{-2x}{(x^2 + 1)\sqrt{x^4 + 2x^2}}$$

$$11. \quad y = \csc^{-1} \sqrt{x} + \sec^{-1} \sqrt{x} = \frac{\pi}{2} \text{ since the secant and cosecant} \\ \text{are cofunctions. } \therefore \frac{dy}{dx} = 0.$$

$$13. \quad y = \cot^{-1} \sqrt{x-1} \Rightarrow \frac{dy}{dx} = \frac{-1}{1 + (\sqrt{x-1})^2} \frac{d}{dx}(\sqrt{x-1}) = \frac{-1}{2x\sqrt{x-1}}$$

$$15. \quad y = \sqrt{x^2 - 4} - 2 \sec^{-1} \frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}}(2x) - 2 \left(\frac{\frac{1}{2}}{|\frac{x}{2}| \sqrt{(\frac{x}{2})^2 - 1}} \right) \\ = \frac{x}{\sqrt{x^2 - 4}} - \frac{1}{|\frac{x}{2}| \sqrt{\frac{x^2 - 4}{4}}} = \frac{x|x| - 4}{|x| \sqrt{x^2 - 4}}$$

$$\text{If } x > 2, |x| = x \text{ and } \frac{dy}{dx} = \frac{\sqrt{x^2 - 4}}{x}. \text{ If } x < -2, |x| = -x \text{ and } \frac{dy}{dx} = \frac{x^2 + 4}{x\sqrt{x^2 - 4}}$$

$$17. \quad y = \tan^{-1} \frac{x-1}{x+1} \Rightarrow \frac{dy}{dx} = \frac{1}{1 + \left(\frac{x-1}{x+1}\right)^2} \frac{d}{dx} \left(\frac{x-1}{x+1} \right) \\ = \left(\frac{(x+1)^2}{(x+1)^2 + (x-1)^2} \right) \left(\frac{x+1 - x+1}{(x+1)^2} \right) = \frac{1}{x^2 + 1}$$

$$19. \quad y = x(\sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2} \sin^{-1} x$$

$$\frac{dy}{dx} = (\sin^{-1} x)^2 + x(2\sin^{-1} x) \left(\frac{1}{\sqrt{1-x^2}} \right) - 2 + 2 \left[\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\sin^{-1} x \right. \\ \left. + 2\sqrt{1-x^2} \left(\frac{1}{\sqrt{1-x^2}} \right) \right] =$$

$$(\sin^{-1} x)^2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{-2x\sin^{-1} x}{\sqrt{1-x^2}} = (\sin^{-1} x)^2.$$

$$21. \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{\frac{1}{2}} = \sin^{-1} \frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}$$

$$23. \int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} |x| \Big|_{\sqrt{2}}^2 = \sec^{-1} 2 - \sec^{-1} \sqrt{2} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

$$25. \int_{-1}^0 \frac{4dx}{1+x^2} = 4 \tan^{-1} x \Big|_{-1}^0 = 0 - 4 \tan^{-1}(-1) = \pi$$

$$27. \int_0^{\frac{\sqrt{2}}{2}} \frac{xdx}{\sqrt{1-x^4}} = \int_0^{\frac{1}{2}} \frac{\frac{1}{2} du}{\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1} u \Big|_0^{\frac{1}{2}} = \frac{\pi}{12}$$

Let $u = x^2 \Rightarrow du = 2xdx$. Then $x = \frac{\sqrt{2}}{2} \Rightarrow u = \frac{1}{2}$; $x = 0 \Rightarrow u = 0$.

$$29. \int_{\frac{1}{\sqrt{3}}}^1 \frac{dx}{x\sqrt{4x^2-1}} = \int_{\frac{2}{\sqrt{3}}}^2 \frac{\frac{1}{2} du}{\frac{1}{2} u \sqrt{u^2-1}} = \sec^{-1} u \Big|_{\frac{2}{\sqrt{3}}}^2 = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

Let $u = 2x \Rightarrow u^2 = 4x^2$ and $du = 2dx$. $x = \frac{1}{\sqrt{3}} \Rightarrow u = \frac{2}{\sqrt{3}}$ and $x = 1 \Rightarrow u = 2$

$$31. \int_0^{\sqrt{2}} \frac{4x dx}{\sqrt{4-x^4}} = 2 \int_0^{\frac{\pi}{2}} du = 2u \Big|_0^{\frac{\pi}{2}} = \pi. \text{ Let } x^2 = 2 \sin u \Rightarrow 2x dx$$

$= 2 \cos u du$. $x = 0 \Rightarrow u = 0$; $x = \sqrt{2} \Rightarrow u = \frac{\pi}{2}$.

$$33. \int_{\sqrt{2}}^{\sqrt[4]{2}} \frac{xdx}{x^2 \sqrt{x^4-1}} = \int_2^{\sqrt{2}} \frac{\frac{1}{2} du}{u \sqrt{u^2-1}} = \frac{1}{2} \sec^{-1} |u| \Big|_2^{\sqrt{2}} = -\frac{\pi}{24}$$

Let $u = x^2 \Rightarrow du = 2xdx$; $x = \sqrt{2} \Rightarrow u = 2$ and $x = \sqrt[4]{2} \Rightarrow u = \sqrt{2}$.

$$35. \int_0^2 \frac{dx}{1+(x-1)^2} = \tan^{-1}(x-1) \Big|_0^2 = \tan^{-1}1 - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$37. \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{dx}{\sqrt{x}\sqrt{1-x}} = \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{2udu}{u\sqrt{1-u^2}} = 2\sin^{-1}u \Big|_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} = \frac{\pi}{6}$$

Let $u = \sqrt{x} \Rightarrow u^2 = x$ and $2udu = dx$. $x = \frac{1}{2} \Rightarrow u = \frac{1}{\sqrt{2}}$ and $x = \frac{3}{4} \Rightarrow u = \frac{\sqrt{3}}{2}$

$$39. \int_{\frac{2}{3}}^{\frac{\sqrt{2}}{3}} \frac{dx}{x\sqrt{9x^2-1}} = \int_{-2}^{-\sqrt{2}} \frac{\frac{1}{3}du}{\frac{1}{3}u\sqrt{u^2-1}} = \sec^{-1}|u| \Big|_{-2}^{-\sqrt{2}} = -\frac{\pi}{12}$$

Let $u = 3x \Rightarrow u^2 = 9x^2$ and $du = 3dx$. $x = -\frac{2}{3} \Rightarrow u = -2$; $x = -\frac{\sqrt{2}}{3} \Rightarrow u = -\sqrt{2}$

$$41. \lim_{x \rightarrow 0} \frac{\sin^{-1}2x}{x} = \lim_{x \rightarrow 0} \frac{\frac{2}{\sqrt{1-4x^2}}}{1} = 2$$

$$43. \lim_{x \rightarrow 0} x^{-3}(\sin^{-1}x - x) = \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{1-x^2}} - 1}{3x^2} = \lim_{x \rightarrow 0} \frac{\frac{x}{\sqrt{1-x^2}}}{6x} = \frac{1}{6}$$

$$45. f(x) = \sin^{-1}x + \cos^{-1}x \equiv \frac{\pi}{2}. \therefore f'(x) = 0 \text{ and } f(.32) = \frac{\pi}{2}.$$

$$47. \tan \theta = x \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt}. \quad x = 2 \Rightarrow \sec \theta = \sqrt{5}.$$

$$(\sqrt{5})^2 \frac{d\theta}{dt} = 4 \Rightarrow \frac{d\theta}{dt} = \frac{4}{5} \text{ rad/hr}$$

49. Yes, $\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x \Rightarrow$ these functions differ by at most a constant.