Important Formulas

1. $\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0 \quad$ (definition of ln x)
2. $\frac{d}{dx}\ln x = \frac{1}{x}$
3. $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$
4. $\int \frac{du}{u} = \begin{cases} \ln u + C & \text{if } u > 0, \\ \ln(-u) + C' & \text{if } u < 0 \end{cases}$
5. $\int \frac{du}{u} = \ln|u| + C \quad$ (if u does not change sign on the domain of integration)

PROBLEMS

In Problems 1–20, find dy/dx.

1. $y = \ln 2x$
2. $y = \ln 5x$
3. $y = \ln kx$ (k constant)
4. $y = (\ln x)^2$
5. $y = \ln(10/x)$
6. $y = \ln(x^2 + 2x)$
7. $y = (\ln x)^3$
8. $y = \ln(\cos x)$
9. $y = \ln(\sec x + \tan x)$
10. $y = x \ln x - x$
11. $y = x^3 \ln(2x)$
12. $y = \ln(\csc x)$
13. $y = \tan^{-1}(\ln x)$
14. $y = \ln(\ln x)$
15. $y = x^2 \ln(x^2)$
16. $y = \ln(x^2 + 4) - x \tan^{-1}\frac{x}{2}$
17. $y = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1}x}{x}$
18. $y = x(\ln x)^3$
19. $y = x[\sin(\ln x) + \cos(\ln x)]$
20. $y = x \ln(a^2 + x^2) - 2x + 2a \tan^{-1}\frac{x}{a}$

Evaluate the integrals in Problems 21–40.

21. $\int \frac{dx}{x}$
22. $\int \frac{2\,dx}{x}$
23. $\int \frac{dx}{2x}$
24. $\int \frac{dx}{x + 2}$
25. $\int_0^1 \frac{dx}{x + 1}$
26. $\int_{-1}^0 \frac{dx}{1 - x}$
27. $\int_{-1}^0 \frac{dx}{2x + 3}$
28. $\int_{-1}^0 \frac{3\,dx}{2 - 3x}$
29. $\int_0^1 \frac{x\,dx}{4x^2 + 1}$
30. $\int_0^\pi \frac{\sin x\,dx}{2 - \cos x}$
31. $\int \tan 3x\,dx$
32. $\int \cot 5x\,dx$
33. $\int \frac{x^2\,dx}{4 - x^3}$
34. $\int \frac{\sec^2 2x\,dx}{1 + \tan 2x}$
35. $\int \frac{dx}{x \ln x}$
36. $\int \frac{dx}{x(\ln x)^2}$
37. $\int_1^2 \frac{(\ln x)^2\,dx}{x}$
38. $\int_1^3 \frac{\cos(\ln x)\,dx}{x}$
39. $\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$
40. $\int \frac{dx}{(1 + x^2)\tan^{-1}x}$

Evaluate the limits in Problems 41–44.

41. $\lim_{x\to\infty} \frac{\ln x}{x}$
42. $\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x}$
43. $\lim_{t\to 0} \frac{\ln(1 + 2t) - 2t}{t^2}$
44. $\lim_{\theta\to 0^+} \frac{\ln(\sin\theta)}{\cot\theta}$
45. Find the area of the "triangular" region in the first qu[illegible] bounded by the lines $x = 1$ and $y = 1$ and the hyp[illegible] $xy = 2$.

and hence $\ln x$ increases without limit as x does. That is,

$$\ln x \to +\infty \quad \text{as} \quad x \to +\infty.$$

On the other hand, as x approaches zero through positive values, $1/x$ tends to plus infinity. Hence, from Eq. (4) we have

$$\ln x = -\ln \frac{1}{x} \to -\infty \quad \text{as} \quad x \to 0^+.$$

The y-axis is a vertical asymptote of the graph of $y = \ln x$.

Properties of $y = \ln x$

1. Domain: The set of all positive real numbers, $x > 0$.
2. Range: The set of all real numbers $-\infty < y < \infty$.
3. It is a continuous, increasing function everywhere on its domain. If $x_1 > x_2 > 0$, then $\ln x_1 > \ln x_2$. It is a one-to-one function from its domain to its range. (It therefore has an inverse, which will be the subject of the next article.)
4. Products, quotients, and powers: If a and x are any two positive numbers, then

$$\ln ax = \ln a + \ln x,$$

$$\ln \frac{x}{a} = \ln x - \ln a,$$

$$\ln x^n = n \ln x.$$

PROBLEMS

Express the logarithms in Problems 1–10 in terms of ln 2 and ln 3. For example, $\ln 1.5 = \ln(3/2) = \ln 3 - \ln 2$.

1. $\ln 16$ **2.** $\ln\sqrt[3]{9}$ **3.** $\ln 2\sqrt{2}$

4. $\ln 0.25$ **5.** $\ln 4/9$ **6.** $\ln 12$

7. $\ln 9/8$ **8.** $\ln 36$ **9.** $\ln 4.5$

10. $\ln\sqrt{13.5}$

In Problems 11–22, find dy/dx.

11. $y = \ln\sqrt{x^2 + 5}$ **12.** $y = \ln x^{3/2}$

13. $y = \ln \dfrac{1}{x\sqrt{x+1}}$ **14.** $y = \ln\sqrt[3]{\cos x}$

15. $y = \ln(\sin x \sin 2x)$ **16.** $y = \ln(x\sqrt{x^2 + 1})$

17. $y = \ln(3x\sqrt{x+2})$ **18.** $y = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$

19. $y = \dfrac{1}{3}\ln\dfrac{x^3}{1+x^3}$ **20.** $y = \ln\dfrac{x}{2+3x}$

21. $y = \ln\dfrac{(x^2+1)^5}{\sqrt{1-x}}$ **22.** $y = \displaystyle\int_{\sqrt{x}}^{\sqrt[3]{x}} \ln t\, dt$

In Problems 23–31, find dy/dx by logarithmic differentiation.

23. $y^2 = x(x+1), \quad x > 0$

24. $y = \sqrt[3]{\dfrac{x+1}{x-1}}, \quad x > 1$

25. $y = \sqrt{x+3}\sin x \cos x, \quad 0 < x < \pi/2$

26. $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}, \quad x > 0$

27. $y = \sqrt[3]{\dfrac{x(x-2)}{x^2+1}}, \quad x > 2$

28. $y^5 = \sqrt{\dfrac{(x+1)^5}{(x+2)^{10}}}$

29. $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}, \quad x > 2$

30. $y^{4/5} = \dfrac{\sqrt{\sin x \cos x}}{1 + 2\ln x}$

31. $\sqrt{y} = \dfrac{x^5 \tan^{-1} x}{(3-2x)\sqrt[3]{x}}$

6.4 THE NATURAL LOGARITHM AND ITS DERIVATIVE

1. $y = \ln 2x \Rightarrow \frac{dy}{dx} = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{2}{2x} = \frac{1}{x}$

3. $y = \ln kx \Rightarrow \frac{dy}{dx} = \frac{1}{kx}\frac{d}{dx}(kx) = \frac{k}{kx} = \frac{1}{x}$

5. $y = \ln\left(\frac{10}{x}\right) = \ln(10x^{-1}) = -\ln 10x \Rightarrow \frac{dy}{dx} = -\frac{1}{x}$

7. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \frac{1}{x} = \frac{3\ln^2 x}{x}$

9. $y = \ln(\sec x + \tan x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sec x + \tan x}(\sec x\tan x + \sec^2 x)$

$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x$

11. $y = x^3 \ln 2x \Rightarrow \frac{dy}{dx} = x^3\left(\frac{2}{2x}\right) + 3x^2 \ln 2x = x^2(1 + 3\ln 2x)$

13. $y = \tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx} = \frac{1}{1 + \ln^2 x}\left(\frac{1}{x}\right) = \frac{1}{x(1 + \ln^2 x)}$

15. $y = x^2 \ln(x^2) \Rightarrow \frac{dy}{dx} = x^2\left(\frac{2x}{x^2}\right) + 2x\ln(x^2) = 2x[1 + \ln(x^2)]$

17. $y = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x}$

$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{2}\left(\frac{2x}{1 + x^2}\right) - \frac{1}{x}\left(\frac{1}{1 + x^2}\right) - (\tan^{-1} x)\left(-\frac{1}{x^2}\right)$

$= \frac{1}{x(1 + x^2)} - \frac{1}{x(1 + x^2)} + \frac{\tan^{-1} x}{x} = \frac{\tan^{-1} x}{x}$

19. $y = x[\sin(\ln x) + \cos(\ln x)]$

$\frac{dy}{dx} = [\sin(\ln x) + \cos(\ln x)] + x\left[\cos(\ln x)\left(\frac{1}{x}\right) - \sin(\ln x)\left(\frac{1}{x}\right)\right]$

$= 2\cos(\ln x)$

21. $\int \frac{dx}{x} = \ln|x| + C$

23. $\int \frac{dx}{2x} = \frac{1}{2}\ln|x| + C$

25. $\int_0^1 \frac{dx}{x+1} = \ln|x+1|\Big]_0^1 = \ln 2 - \ln 1 = \ln 2$

27. $\int_{-1}^0 \frac{dx}{2x+3} = \frac{1}{2}\ln|2x+3|\Big]_{-1}^0 = \frac{1}{2}[\ln 3 - \ln 1] = \frac{1}{2}\ln 3$

29. $\int_0^1 \frac{x\,dx}{4x^2+1} = \frac{1}{8}\ln(4x^2+1)\Big]_0^1 = \frac{1}{8}\ln 5$

31. $\int \tan 3x\,dx = \int \frac{\sin 3x}{\cos 3x}dx = -\frac{1}{3}\ln|\cos 3x| + C$

33. $\int \frac{x^2\,dx}{4-x^3} = -\frac{1}{3}\ln|4-x^3| + C$

35. $\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$

37. $\int_1^2 \frac{(\ln x)^2}{x}dx = \int_0^{\ln 2} u^2\,du = \frac{1}{3}u^3\Big]_0^{\ln 2} = \frac{1}{3}\ln^3 2$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx;\ x = 1 \Rightarrow u = \ln 1 = 0;\ x = 2 \Rightarrow u = \ln 2$

39. $\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx = \int u^{-1}du = \ln|u| + C = \ln|\sec x + \tan x| + C$

Let $u = \sec x + \tan x \Rightarrow du = \sec x \tan x + \sec^2 x\,dx$

41. $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right) = 0$

43. $\lim_{t\to 0} \frac{\ln(1+2t) - 2t}{t^2} = \lim_{t\to 0}\left(\frac{\frac{2}{1+2t} - 2}{2t}\right) = \lim_{t\to 0}\left(\frac{\frac{-4}{(1+2t)^2}}{2}\right) = -2$

45. $A = \int_1^2 \left(\frac{2}{x} - 1\right)dx = 2\ln|x| - x\Big]_1^2 = 2\ln 2 - 1$

47. $A=\int_0^1 \frac{2}{1+x^2}\,dx = 2\tan^{-1}x]_0^1 = \frac{\pi}{2}$

$\bar{x}=\frac{2}{\pi}\int_0^1 x\left(\frac{2}{1+x^2}\right)dx=\frac{2}{\pi}\ln|1+x^2|\Big]_0^1=\frac{2\ln 2}{\pi}$

$\bar{y}=0$ by symmetry. $\therefore\ (\bar{x},\bar{y})=\left(\frac{2\ln 2}{\pi},0\right)$.

49. $x=\ln(\sec t+\tan t)-\sin t \Rightarrow \frac{dx}{dt}=\frac{\sec t\tan t+\sec^2 t}{\sec t+\tan t}-\cos t$

$\left(\frac{dx}{dt}\right)^2=(\sec t-\cos t)^2=\sec^2 t-2+\cos^2 t.$

$y=\cos t \Rightarrow \frac{dy}{dt}=-\sin t \Rightarrow \left(\frac{dy}{dt}\right)^2=\sin^2 t$

$s=\int_0^{\frac{\pi}{3}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{\frac{\pi}{3}}\sqrt{\sec^2 t-2+\cos^2 t+\sin^2 t}\,dt$

$\int_0^{\frac{\pi}{3}}\sqrt{\sec^2 t-1}\,dt=\int_0^{\frac{\pi}{3}}\tan t\,dt=\int_0^{\frac{\pi}{3}}\frac{\sin t}{\cos t}\,dt=$

$-\ln|\cos t|\Big]_0^{\frac{\pi}{3}}=-\left[\ln\frac{1}{2}-\ln 1\right]=-\ln\frac{1}{2}=\ln 2$

6.5 PROPERTIES OF THE NATURAL LOGARITHM

1. $\ln 16=\ln 2^4=4\ln 2$

3. $\ln 2\sqrt{2}=\ln 2^{\frac{3}{2}}=\frac{3}{2}\ln 2$

5. $\ln\frac{4}{9}=\ln 4-\ln 9=2\ln 2-2\ln 3$

7. $\ln\frac{9}{8}=\ln 9-\ln 8=2\ln 3-3\ln 2$

9. $\ln 4.5=\ln\frac{9}{2}=\ln 9-\ln 2=2\ln 3-\ln 2$

11. $y=\ln\sqrt{x^2+5}=\frac{1}{2}\ln(x^2+5) \quad\Rightarrow\quad \frac{dy}{dx}=\frac{1}{2}\left(\frac{2x}{x^2+5}\right)=\frac{x}{x^2+5}$

13. $y=\ln\frac{1}{x\sqrt{x+1}}=-\ln x\sqrt{x+1}=-(\ln x+\frac{1}{2}\ln(x^2+5)$

$$\frac{dy}{dx} = -\left[\frac{1}{x} + \frac{1}{2(x+1)}\right]$$

15. $y = \ln(\sin x \sin 2x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sin x \sin 2x} \cdot \frac{d}{dx}(\sin x \sin 2x)$

$$= \frac{\sin x(2\cos x) + (\sin 2x)\cos x}{\sin x \sin 2x}$$

$$= \frac{2\cos 2x + 2\cos^2 x}{\sin 2x}$$

17. $y = \ln(3x\sqrt{x+2}) = \ln 3x + \frac{1}{2}\ln(x+2)$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(x+2)}$$

19. $y = \frac{1}{3}\ln\left(\frac{x^3}{1+x^3}\right)$

$$= \frac{1}{3}\left[\ln x^3 - \ln(1+x^3)\right]$$

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{3x^2}{x^3} - \frac{3x^2}{1+x^3}\right]$$

21. $y = \ln\frac{(x^2+1)^5}{\sqrt{1-x}} = 5\ln(x^2+1) - \frac{1}{2}\ln(1-x)$

$$\frac{dy}{dx} = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$$

23. $y^2 = x(x+1),\ x > 0$

$$2\ln y = \ln x + \ln(x+1)$$

$$\frac{2}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x+1} \Rightarrow \frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right)$$

25. $y = \sqrt{x+2}\sin x\cos x,\ 0 < x < \frac{\pi}{2} \Rightarrow \ln y = \frac{1}{2}\ln(x+2) + \ln\sin x + \ln\cos x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(x+2)} + \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = y\left[\frac{1}{2(x+2)} + \cot x - \tan x\right]$$

27. $y=\sqrt[3]{\dfrac{x(x-2)}{x^2+1}}$, $x>2 \Rightarrow \ln y=\frac{1}{3}[\ln x+\ln(x-2)-\ln(x^2+1)]$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{3}\left[\frac{1}{x}+\frac{1}{x-2}-\frac{2x}{x^2+1}\right] \Rightarrow \frac{dy}{dx}=\frac{y}{3}\left[\frac{1}{x}+\frac{1}{x-2}-\frac{2x}{x^2+1}\right]$$

29. $y=\sqrt{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$, $x>2$

$$\ln y=\frac{1}{3}[\ln x+\ln(x+1)+\ln(x-2)-\ln(x^2+1)-\ln(2x+3)]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{3}\left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x-2}-\frac{2x}{x^2+1}-\frac{2}{2x+3}\right]$$

$$\frac{dy}{dx}=\frac{y}{3}\left[\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x-2}-\frac{2x}{x^2+1}-\frac{2}{2x+3}\right]$$

31. $\sqrt{y}=\dfrac{x^5\tan^{-1}x}{(3-2x)\sqrt[3]{x}} \Rightarrow$

$$\frac{1}{2}y=5\ln x+\ln\tan^{-1}x-\ln(3-2x)-\frac{1}{3}\ln x$$

$$\frac{1}{2y}\frac{dy}{dx}=\frac{5}{x}+\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}+\frac{2}{3-2x}-\frac{1}{3x}$$

$$\frac{dy}{dx}=2y\left[\frac{14}{3x}+\frac{2}{3-2x}+\frac{1}{(1+x^2)\tan^{-1}x}\right]$$

33. $\displaystyle\int_{-1}^{1}\frac{dx}{x+3}=\ln|x+3|\Big]_{-1}^{1}=\ln 4-\ln 2=\ln 2$

35. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot x\,dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos x}{\sin x}\,dx=\ln|\sin x|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\ln\sqrt{2}$

37. $\displaystyle\int_{2}^{4}\frac{2x-5}{x}\,dx=\int_{2}^{4}\left(2-\frac{5}{x}\right)dx=2x-5\ln|x|\Big]_{2}^{4}=4-5\ln 2$

39. (a) $\displaystyle\int_{0}^{\frac{3}{5}}\frac{x\,dx}{1-x^2}=-\frac{1}{2}\int_{1}^{\frac{16}{25}}\frac{du}{u}=-\frac{1}{2}\ln|u|\Big]_{1}^{\frac{16}{25}}=\ln\left(\frac{16}{25}\right)^{-1/2}=\ln\frac{5}{4}$

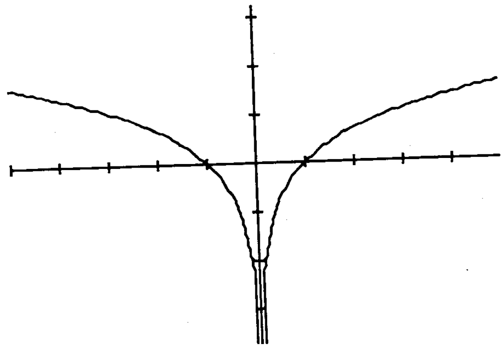
Let $u=1-x^2 \Rightarrow du=-2x\,dx \Rightarrow x\,dx=-\frac{1}{2}u$;

$x=0 \Rightarrow u=1$; $x=\frac{3}{5} \Rightarrow u=\frac{16}{25}$

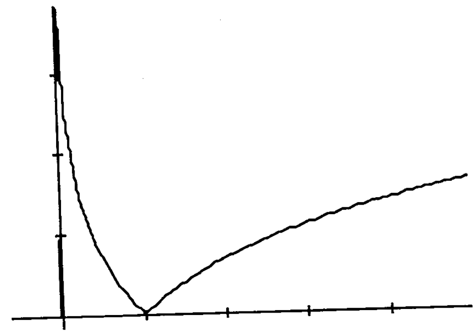
(b) $\displaystyle\int_0^{\frac{3}{5}} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big]_0^{\frac{3}{5}} = \sin^{-1}\frac{3}{5}$

(c) $\displaystyle\int_0^{\frac{3}{5}} \frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\,\Big]_0^{\frac{3}{5}} = \frac{1}{5}$

41. (a) $y = \log|x|$



(b) $y = |\ln x|$



43. $\displaystyle\lim_{x\to\infty}\int_x^{2x} \frac{1}{t}\,dt = \lim_{x\to\infty}[\ln t\,\Big]_x^{2x} = \lim_{x\to\infty}[\ln 2x - \ln x] = \ln 2$

45. The region is bounded by $y = \dfrac{4}{x}$ and $y = (x-3)^2$. These curves intersect where $\dfrac{4}{x} = (x-3)^2 \Leftrightarrow x^3 - 6x^2 + 9x - 4 = 0 \Leftrightarrow (x-1)^2\,(x-4) = 0$

or $x = 1, 4$. Then:

$$A = \int_1^4 \left[\frac{4}{x} - (x-3)^2\right]dx = 4\ln|x| - \frac{1}{3}(x-3)^3\Big]_1^4 = 4\ln 4 - 3$$

$$V = \pi\int_1^4 \left[\left(\frac{4}{x}\right)^2 - (x-3)^4\right]dx = \pi\left[-\frac{16}{x} - \frac{1}{5}(x-3)^5\right]_0^4 = \frac{27\pi}{5}$$

47. (a) If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, $f''(x) = -\dfrac{1}{(1+x)^2}$ and

$f'''(x) = \dfrac{2}{(1+x)^3}$. Then $L(x) \approx f(0) + f'(0)x = \ln 1 + (1)\,x = x$.

$Q(x) \approx f(0) + f'(0)\,x + \dfrac{f''(0)}{2}x^2 = \ln 1 + (1)x - \dfrac{1}{2}x^2 = x - \dfrac{x^2}{2}$.

(b) $|e_1(x)| \le \frac{1}{2}M_1x^2$, where $M_1 = \max\{|f''(x)| : 0 \le x \le .1\} = 1$

$\therefore\ |e_1(x)| \le \frac{1}{2}(1)(.1)^2 = 0.005$

$|e_2(x)| \le \frac{1}{6}M_2\,|x|^3$, where $M_2 = \max\{|f'''(x)| : 0 \le x \le .1\} = 2$.

$\therefore\ |e_2(x)| \le \frac{1}{6}(2)(.1)^3 = 0.000\overline{3}$

49. (a) $\ln 1.2 = \ln(1 + 0.2) \approx 0.2$ by (16)

$$\approx 0.2 - \frac{(.2)^2}{2} = 0.1800 \text{ by } (17)$$

$$\ln 1.2 \approx \frac{.1}{3}\left[1 + 4\left(\frac{1}{1.1}\right) + \frac{1}{1.2}\right] \approx 0.18232 \text{ by Simpson's Rule.}$$

(b) $\ln .8 = \ln(1 - 0.2) \approx -0.2$ by (16)

$$\approx -0.2 - \frac{(-.2)^2}{2} = -0.2200 \text{ by } (17)$$

$$\ln .8 \approx -\frac{.1}{3}\left[\frac{1}{.8} + 4\left(\frac{1}{.9}\right) + 1\right] \approx -0.22315 \text{ by Simpson's Rule.}$$