PROBLEMS

Find the critical points for each of the functions in Problems 1–30. For each critical point, determine whether the function has a local maximum, a local minimum, or neither there. If possible, find the absolute maximum and minimum values of the function on the indicated domain.

1. $y = x - x^2$ on $[0, 1]$
2. $y = x - x^3$, $-\infty < x < \infty$
3. $y = x - x^3$ on $[0, 1]$
4. $y = x^3 - 3x^2 + 2$, $-\infty < x < \infty$
5. $y = x^3 - 147x$, $-\infty < x < \infty$
6. $y = x^3 - 2x^2 + x$ on $[-1, 2]$
7. $y = x^2 - 4x + 3$ on $(0, 3)$
8. $y = x^3 - 6x$ on $[0, 2]$
9. $y = x - x^2$ on $(0, 1)$
10. $y = 1/(x - 2)$ on $(1, 3)$
11. $y = 2x$ on $[0, 3]$
12. $y = 1/(3 - x)$ on $[0, 4]$
13. $y = x^2 + (2/x)$, $x > 0$
14. $y = x/(1 + x)$ on $[0, 1]$
15. $y = x^3 + 3x^2 + 3x + 2$, $-\infty < x < \infty$
16. $y = -x^2 + 4x$, $x \geq 0$
17. $y = \sqrt{x} - x$, $x \geq 0$
18. $y = \sqrt{4 - x^2}$, $-2 \leq x \leq 2$
19. $y = x^4 - 4x$ on $[0, 2]$
20. $y = x^4 - x^2$ on $[-1, 1]$
21. $y = \tan x$ on $[0, \pi/2)$
22. $y = \sec x$ on $(-\pi/2, \pi/2)$
23. $y = 2 \sin x + \cos 2x$ on $[0, \pi/2]$
24. $y = x^4 - 2x^2 + 2$ on $[-1, 2]$
25. $y = x^4 - 8x^3 - 270x^2$, $-\infty < x < \infty$
26. $y = x^4 - (x^3/3) - 2x^2 + x - 1$, $-\infty < x < \infty$
27. $y = (x - x^2)^{-1}$ on $(0, 1)$
28. $y = |x^3|$ on $[-2, 3]$. What would occur if the domain were changed to $[-2, 3)$?
29. $y = \begin{cases} -x & \text{for } x \leq 0, \\ 2x - x^2 & \text{for } x > 0 \end{cases}$
30. $y = \begin{cases} 3 - x \text{ on } [0, 2], \\ (1/2)x^2 \text{ on } (2, 3] \end{cases}$

Find the absolute maximum and minimum values (if any) of the functions in Problems 31–36. In each case, start by rewriting the given formula without absolute values by using appropriate versions of the formula on different intervals of the domain. Then include the boundaries of these intervals among the points you investigate. Unless stated otherwise, the domain is the function's natural (largest possible) domain.

31. $y = \dfrac{x}{1 + |x|}$
32. $y = \dfrac{|x|}{1 + |x|}$
33. $y = \sin|x|$, $-2\pi \leq x \leq 2\pi$
34. $y = \dfrac{|x|}{x}$
35. $y = |x^2 - 1|$, $-1 \leq x \leq 2$
36. $y = |x - x^2|$, $x \geq 0$
37. We do not need calculus to show that $x + 1/x \geq 2$ when $x > 0$. To see why, multiply out the left side of the inequality $(x - 1)^2 \geq 0$ and divide through by x.
38. Calculate the values of the first and second derivatives of $y = x^3$, $y = x^4$, and $y = -x^4$ at the origin to show that y'' has no predictive value when it is zero.
39. Find the critical points, asymptotes, and points of inflection and graph the function
$$y = \frac{x}{x^2 + 1}.$$
40. Test the function
$$y = \frac{x^3}{6} + \frac{x^2}{2} - 1 + \cos x$$
for the existence of a local maximum or minimum at $x = 0$.
41. Suppose that a function $y = f(x)$ is known to be differentiable for all values of x and to have a local maximum at $x = c$. Which of the following must be true of the graph of f'?
    a) It has a point of inflection at $x = c$.
    b) It crosses the x-axis at $x = c$.
    c) It has a local maximum or minimum at $x = c$.
42. Find the maximum height of the curve $y = 4 \sin x - 3 \cos x$ above the x-axis.
43. Find the maximum height of the curve $y = 4 \sin^2 x - 3 \cos^2 x$ above the x-axis.
44. a) Find a value of b that will ensure that the function $y = 2x^3 + bx + c$ has a local minimum value at $x = 1$.
    b) Why will no value of b make $y = 2x^3 + bx + c$ have a local maximum value at $x = 1$?

3.4 MAXIMA AND MINIMA: THEORY

1. $y = x - x^2$ on $[0,1]$. $y' = 1 - 2x$. $y' = 0 \Leftrightarrow x = \frac{1}{2}$. $y' > 0$ if $x < \frac{1}{2}$ and $y' < 0$ if $x > \frac{1}{2} \Rightarrow y\left(\frac{1}{2}\right) = \frac{1}{4}$ is local maximum which absolute, since $y(0) = 0$ and $y(1) = 0$ are absolute minimimum values.

3. $y = x - x^3$ on $[0,1]$. $y' = 1 - 3x^2$. $y' = 0 \Leftrightarrow x = \pm \frac{1}{\sqrt{3}}$. The value $x = -\frac{1}{\sqrt{3}}$ is out of the domain. $y\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9}$ is local maximum which is absolute, since $y(0) = 0$ and $y(1) = 0$ are absolute minimum values.

5. $y = x^3 - 147x$ on $(-\infty, \infty)$. $y' = 3x^2 - 147$. $y' = 0 \Leftrightarrow x = \pm 7$. $y' > 0$ if $x < -7$, $y' < 0$ if $-7 < x < 7$, and $y' > 0$ if $x > 7$. Therefore, $y(-7) = 686$ is local maximum, and $y(7) = -686$ is local minimum. There are no absolute extrema.

7. $y = x^2 - 4x + 3$ on $(0, 3)$. $y' = 2x - 4$. $y' = 0 \Leftrightarrow x = 2$. $y'' = 2$ $\Rightarrow y(2) = -1$ is absolute minimum. No absolute maximum

values since the domain interval is open.

9. $y = x - x^2$ on $(0, 1)$. $y' = 1 - 2x$. $y' = 0 \Leftrightarrow x = \frac{1}{2}$. $y'' = -2 \Rightarrow$ $y\left(\frac{1}{2}\right) = \frac{1}{4}$ is a local maximum which is absolute. There are no other extrema since the domain interval is open.

11. $y = 2x$ on $[0, 3]$. $y' = 2 \neq 0 \Rightarrow$ no local extrema. $y(0) = 0$ is absolute minimun and $y(3) = 6$ is absolute maximum.

13. $y = x^2 + \frac{2}{x}$, $x > 0$. $y' = 2x - \frac{2}{x^2}$. $y' = 0 \Leftrightarrow 2x^3 - 2 = 0 \Leftrightarrow x = 1$. $y'' = 2 + \frac{4}{x^3} > 0$ if $x = 1 \Rightarrow y(1) = 3$ is local minimum which is absolute on $x > 0$. There are no other extreme values.

15. $y = x^3 + 3x^2 + 3x + 2$, $-\infty < x < \infty$. $y' = 3x^2 + 6x + 3$. $y' = 0 \Leftrightarrow$ $3(x+1)^2 = 0 \Leftrightarrow x = -1$. There is no sign change in y'. Therefore there are no extreme values.

17. $y = \sqrt{x} - x$, $x \geq 0$. $y' = \frac{1}{2\sqrt{x}} - 1$. $y' = 0 \Leftrightarrow 1 - 2\sqrt{x} = 0 \Leftrightarrow x = \frac{1}{4}$.

$y'' = -\frac{1}{4}x^{-\frac{3}{2}} < 0 \Rightarrow y\left(\frac{1}{4}\right) = \frac{1}{4}$ is local maximum which is absolute. There is no local minimum.

19. $y = x^4 - 4x$ on $[0,2]$. $y' = 4x^3 - 4$. $y' = 0 \Leftrightarrow x = 1$.

$y'' = 12x^2$. $y''(1) > 0 \Rightarrow y(1) = -3$ is a relative minimum. $y(0) = 0$ and $y(2) = 8$. Therefore 8 is the absolute maximum, and -3 is an absolute minimum.

21. $y = \tan x$ on $\left[0, \frac{\pi}{2}\right]$. $y' = \sec^2 x > 0$. Therefore $y = \tan x$ is always rising; $y(0) = 0$ is absolute minimum, and there is no maximum.

23. $y = 2\sin x + \cos 2x$ on $\left[0, \frac{\pi}{2}\right]$. $y' = 2\cos x - 2\sin 2x$

$= 2\cos x - 4\sin x\cos x = 2\cos x(1 - 2\sin x)$. $y' = 0 \Leftrightarrow \cos x = 0$ or $\sin x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{2}$ or $\frac{\pi}{6}$. $y'' = -2\sin x - 4\cos 2x$. $y''\left(\frac{\pi}{6}\right) =$ $-2\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right) < 0 \Rightarrow y\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{2}\right) + \frac{1}{2} = \frac{3}{2} =$ local maximum. $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = 1$ are absolute minima, and the local maximum becomes absolute.

25. $y = x^4 - 8x^3 - 270x^2$ on $-\infty < x < \infty$. $y' = 4x^3 - 24x^2 - 540x$. $y' = 0 \Leftrightarrow$ $4x(x - 15)(x + 9) = 0 \Leftrightarrow x = -9, 0, 15$. $y(-9) = -9477$ is a local minimum and $y(15) = -37125$ is an absolute minimum. $y(0) = 0$ is a local maximum.

27. $y = (x - x^2)^{-1}$ on $(0,1)$. $y' = -\dfrac{1 - 2x}{(x - x^2)^2}$. $y' = 0 \Leftrightarrow x = \frac{1}{2}$.

$y' < 0$ if $x < \frac{1}{2}$ and $y' > 0$ if $x > \frac{1}{2} \Rightarrow y\left(\frac{1}{2}\right) = 4$ is a local minimum which is absolute on $(0,1)$. $y \to \infty$ if $x \to 0^-$ or $x \to 1^+ \Rightarrow$ no maximum values.

31. $y = \dfrac{x}{1+|x|}$ can be expressed as

$$y = \begin{cases} \dfrac{x}{1-x} & \text{for } x \le 0 \\ \dfrac{x}{1+x} & \text{for } x > 0 \end{cases} \quad \text{Then } y' = \begin{cases} \dfrac{1}{(1-x)^2} & \text{for } x \le 0 \\ \dfrac{1}{(1+x)^2} & \text{for } x > 0 \end{cases}$$

Note that each point for which y' does not exist is not in the domain of that piece. Therefore, $y' > 0$ always $\Rightarrow$ there are no extreme values.

33. $y = \sin|x|$, $-2\pi \le x \le 2\pi$.

$$y = \begin{Bmatrix} \sin x & \text{for } 0 \le x \le 2\pi \\ -\sin x & -2\pi \le x < 0 \end{Bmatrix} \quad y' = \begin{Bmatrix} \cos x & \text{for } 0 < x \le 2\pi \\ -\cos x & \text{for } -2\pi \le x < 0 \end{Bmatrix}$$

$x \to 0^+ \Rightarrow y' \to 1$ and $x \to 0^- \Rightarrow y' \to -1$. Therefore, $y'(0)$ does not exist and 0 is a critical point of y. $\cos x = 0$ for $x = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$ in $(0, 2\pi]$ and $-\cos x = 0$ for

$x = -\dfrac{\pi}{2}$ and $-\dfrac{3\pi}{2}$ in $[-2\pi, 0)$. $y(0) = 0$ is local minimum.

$y\left(\pm\dfrac{\pi}{2}\right) = 1$ are absolute maximum values. $y\left(\pm\dfrac{3\pi}{2}\right) = -1$

are absolute minimum values.

35. $y = |x^2 - 1|$, $-1 \le x \le 2$.

$$y = \begin{Bmatrix} 1 - x^2 & \text{for } -1 \le x \le 1 \\ x^2 - 1 & \text{for } 1 < x \le 2 \end{Bmatrix} \quad y' = \begin{Bmatrix} -2x & \text{for } -1 < x < 1 \\ 2x & \text{for } 1 < x \le 2 \end{Bmatrix}$$

$y' = 0 \Leftrightarrow -2x = 0 \Leftrightarrow x = 0$. $y'' = -2 < 0 \Rightarrow y(0) = 1$ is local maximum.

y' does not exist if $x = 1$. $y(-1) = 0$, $y(1) = 0$ are absolute minimum values. $y(2) = 3$ is absolute maximum.

37. $(x-1)^2 \ge 0 \Leftrightarrow x^2 - 2x + 1 \ge 0 \Leftrightarrow x^2 + 1 \ge 2x$. If $x > 0$, then

$x + \dfrac{1}{x} \ge 2$