Derivatives of the Basic Trigonometric Functions

If u is a differentiable function of x, then:

1. $\dfrac{d}{dx}\sin u = \cos u\dfrac{du}{dx}$,
2. $\dfrac{d}{dx}\cos u = -\sin u\dfrac{du}{dx}$,
3. $\dfrac{d}{dx}\tan u = \sec^2 u\dfrac{du}{dx}$,
4. $\dfrac{d}{dx}\sec u = \sec u \tan u\dfrac{du}{dx}$,
5. $\dfrac{d}{dx}\csc u = -\csc u \cot u\dfrac{du}{dx}$,
6. $\dfrac{d}{dx}\cot u = -\csc^2 u\dfrac{du}{dx}$.

PROBLEMS

In Problems 1–36, find dy/dx.

1. $y = \sin(x + 1)$
2. $y = -\cos x$
3. $y = \sin(x/2)$
4. $y = \sin(-x)$
5. $y = \cos 5x$
6. $y = \cos(-x)$
7. $y = \cos(-2x)$
8. $y = \sin 7x$
9. $y = \sin(3x + 4)$
10. $y = \cos(2 - x)$
11. $y = x \sin x$
12. $y = \sin 5(x - 1)$
13. $y = x \sin x + \cos x$
14. $y = \dfrac{1}{\sin x}$
15. $y = \dfrac{1}{\cos x}$
16. $y = \dfrac{\sin x}{\cos x}$
17. $y = \sec(x - 1)$
18. $y = \cot(-x)$
19. $y = \sec(1 - x)$
20. $y = \dfrac{2}{\cos 3x}$
21. $y = \tan 2x$
22. $y = \cos(ax + b)$
23. $y = \sin^2 x$
24. $y = \sin^2 x + \cos^2 x$
25. $y = \cos^2 5x$
26. $y = \cot^2 x$
27. $y = \tan(5x - 1)$
28. $y = \sin x - x \cos x$
29. $y = 2 \sin x \cos x$
30. $y = \sec(x^2 + 1)$
31. $y = \sqrt{2 + \cos 2x}$
32. $y = \sin(1 - x^2)$
33. $y = \cos\sqrt{x}$
34. $y = \sec^2 x - \tan^2 x$
35. $y = \sqrt{\dfrac{1 + \cos 2x}{2}}$ (*Hint:* Use a half-angle formula first.)
36. $y = \sin^2 x^2$

Assume that each of the equations in Problems 37–41 defines y as a differentiable function of x. Find dy/dx by implicit differentiation.

37. $x = \tan y$
38. $x = \sin y$
39. $y^2 = \sin^4 2x + \cos^4 2x$
40. $x + \sin y = xy$
41. $x + \tan(xy) = 0$

42. Assume that the equation $2xy + \pi \sin y = 2\pi$ defines y as a differentiable function of x. Find dy/dx when $x = 1$ and $y = \pi/2$.

43. Find an equation for the tangent to the curve $x \sin 2y = y \cos 2x$ at the point $(\pi/4, \pi/2)$.

Find the limits in Problems 44–51.

44. $\lim_{x\to 2} \sin\left(\dfrac{1}{x} - \dfrac{1}{2}\right)$
45. $\lim_{x\to \pi/4} \dfrac{\sin x}{\cos x}$
46. $\lim_{x\to -\pi} \cos^2 x$
47. $\lim_{x\to \pi} \sec(1 + \cos x)$
48. $\lim_{x\to 0} (\sec x + \tan x)$
49. $\lim_{x\to 0} x \csc x$
50. $\lim_{h\to 0} \dfrac{\sin(a + h) - \sin a}{h}$
51. $\lim_{h\to 0} \dfrac{\cos(a + h) + \cos a}{h}$

52. Find an equation for the tangent to the curve $y = \sin mx$ at $x = 0$.

53. Graph $y = \tan x$ and its linearization $y = x$ together for $-\pi/4 \le x \le \pi/4$.

35. $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \bullet \frac{\frac{1}{\cos A\cos B}}{\frac{1}{\cos A\cos B}}$

$$\frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \tan(A+(-B)) = \frac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)} = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

2.7 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. $y = \sin(x + 1) \Rightarrow y' = \cos(x + 1)$

3. $y = \sin\left(\frac{x}{2}\right) \Rightarrow y' = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

5. $y = \cos 5x \Rightarrow y' = -5\sin 5x$

7. $y = \cos(-2x) = \cos(2x) \Rightarrow y' = -\sin(2x)(2) = -2\sin(2x)$

9. $y = \sin(3x + 4) \Rightarrow y' = 3\cos(3x + 4)$

11. $y = x\sin x \Rightarrow y' = x\cos x + \sin x$

13. $y = x\sin x + \cos x \Rightarrow y' = x\cos x + \sin x - \sin x = x\cos x$

15. $y = \frac{1}{\cos x} = \sec x \Rightarrow y' = \sec x\tan x$

17. $y = \sec(x - 1) \Rightarrow y' = \sec(x - 1)\tan(x - 1)$

19. $y = \sec(1 - x) \Rightarrow y' = -\sec(1 - x)\tan(1 - x)$

21. $y = \tan 2x \Rightarrow y' = 2\sec^2 2x$

23. $y = \sin^2 x \Rightarrow y' = 2\sin x\cos x = \sin 2x$

25. $y = \cos^2 5x \Rightarrow y' = -10\cos 5x\sin 5x$ $(=-5\sin 10x)$

27. $y = \tan(5x-1) \Rightarrow y' = 5\sec^2(5x-1)$

29. $y = 2\sin x\cos x = \sin 2x \Rightarrow y' = 2\cos 2x$

31. $y = \sqrt{2 + \cos 2x} \Rightarrow y' = \frac{1}{2}(2 + \cos 2x)^{-\frac{1}{2}}(-2\sin 2x) = \frac{-\sin 2x}{\sqrt{2 + \cos 2x}}$

33. $y = \cos\sqrt{x} \Rightarrow \frac{dy}{dx} = -\sin\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right) = -\frac{\sin\sqrt{x}}{2\sqrt{x}}$

35. $y = \sqrt{\frac{1+\cos 2x}{2}} = \sqrt{\cos^2 x} = |\cos x| \Rightarrow \frac{dy}{dx} = -\sin x$ if $\cos x > 0$ and $\sin x$ if $\cos x < 0$.

37. $x = \tan y \Rightarrow 1 = \sec^2 y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

39. $y^2 = \sin^4 2x + \cos^4 2x \Rightarrow$

$$2y\frac{dy}{dx} = 4(\sin^3 2x)(\cos 2x)(2) + 4(\cos^3 2x)(-\sin 2x)(2)$$

$$= 8\sin 2x\cos 2x(\sin^2 2x - \cos^2 2x) = 4\sin 4x(-\cos 4x) = -2\sin 8x$$

$$\frac{dy}{dx} = -\frac{\sin 8x}{y}$$

41. $x + \tan(xy) = 0 \Rightarrow 1 + \sec^2(xy)\left[y + x\frac{dy}{dx}\right] = 0$

$$1 + y\sec^2(xy) = -x\sec^2(xy)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-(1 + y\sec^2(xy))}{x\sec^2(xy)}$$

or $\frac{dy}{dx} = -\frac{1}{x}\left[\cos^2(xy) + y\right]$

43. $x\sin 2y = y\cos 2x \Rightarrow 2x\cos 2y\frac{dy}{dx} + \sin 2y = -2y\sin 2x + \cos 2x\frac{dy}{dx}$

Evaluating at $(\frac{\pi}{4}, \frac{\pi}{2})$:

$$2(\frac{\pi}{4})(\cos\pi)\frac{dy}{dx} + \sin\pi = -2(\frac{\pi}{2})\sin\frac{\pi}{2} + (\cos\frac{\pi}{2})\frac{dy}{dx}$$

$$-\frac{\pi}{2}\frac{dy}{dx} = -\pi \Rightarrow \frac{dy}{dx} = 2$$

$y - \frac{\pi}{2} = 2(x - \frac{\pi}{4}) \Rightarrow y = 2x$ is the equation of the tangent.

45. $\lim_{x\to\frac{\pi}{4}} \frac{\sin x}{\cos x} = \lim_{x\to\frac{\pi}{4}} \tan x = 1$

47. $\lim_{x\to\pi} \sec(1+\cos x) = \sec(1+\cos\pi) = \sec(1-1) = \sec 0 = 1$

49. $\lim_{x\to 0} x\csc x = \lim_{x\to 0} \frac{x}{\sin x} = 1$