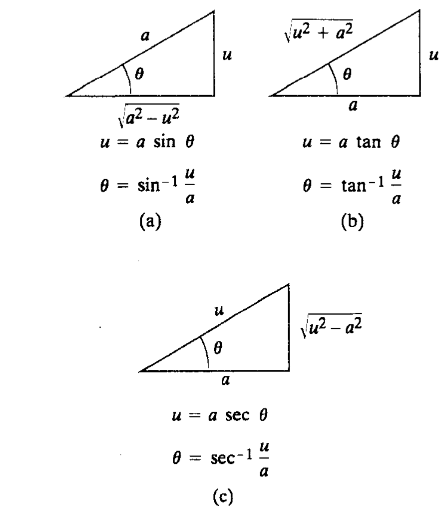
PROBLEMS



Reference triangles.

In Problems 1–15, evaluate the integrals.

1. $\int_{-2}^{2} \frac{dx}{4+x^2}$

2. $\int_{0}^{2} \frac{dx}{8+2x^2}$

3. $\int \frac{dx}{1+4x^2}$

4. $\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

5. $\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

6. $\int_{0}^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$

7. $\int \frac{dy}{\sqrt{25+y^2}}$

8. $\int \frac{3\,dy}{\sqrt{1+9y^2}}$

9. $\int \frac{dy}{\sqrt{25+9y^2}}$

10. $\int \frac{dz}{\sqrt{z^2-4}}$

11. $\int \frac{3\,dz}{\sqrt{9z^2-1}}$

12. $\int \frac{dz}{\sqrt{25z^2-9}}$

13. $\int_{8/\sqrt{3}}^{8} \frac{dx}{x\sqrt{x^2-16}}$

14. $\int_{2}^{\sqrt{6}} \frac{dx}{x\sqrt{x^2-3}}$

15. $\int_{\sqrt{2}/2}^{1} \frac{dx}{2x\sqrt{4x^2-1}}$

In Problems 16–22, evaluate the integrals with the substitutions given.

16. $\int \frac{dy}{y^2\sqrt{y^2-16}}, \quad y = 4\sec u$

17. $\int_{2}^{4} \sqrt{x^2-4}\,dx, \quad x = 2\sec u$

18. $\int_{0}^{4/5} \frac{x^3\,dx}{\sqrt{1-x^2}}, \quad x = \cos u$

19. $\int \frac{dx}{x^2\sqrt{9-x^2}}, \quad x = 3\sin u$

20. $\int_{0}^{1/2} \frac{dx}{\sqrt{1+x^2}}, \quad x = \tan u$

21. $\int_{5/4}^{5/3} \frac{dx}{x^2\sqrt{x^2-1}}, \quad x = \csc u$

22. $\int_{1/2}^{1} \frac{\sqrt{1-x}}{x}\,dx, \quad x = \cos^2 u$

Evaluate the integrals in Problems 23–38.

23. $\int_{0}^{5} \sqrt{25-x^2}\,dx$

24. $\int \frac{dx}{\sqrt{1-4x^2}}$

25. $\int_{1}^{2} \frac{dx}{\sqrt{4-(x-1)^2}}$

26. $\int_{0}^{2} \frac{dx}{\sqrt{4+x^2}}$

27. $\int_{0}^{1} \frac{12\,dx}{\sqrt{4-x^2}}$

28. $\int_{0}^{3/2} \frac{x\,dx}{\sqrt{4+x^2}}$

29. $\int_{0}^{1} \frac{x^3\,dx}{\sqrt{x^2+1}}$

30. $\int \frac{x+1}{\sqrt{4-x^2}}\,dx$

31. $\int \frac{dx}{x\sqrt{x^2-1/4}}$

32. $\int \frac{dx}{\sqrt{2-5x^2}}$

33. $\int \frac{\sqrt{1-x^2}}{x^2}\,dx$

34. $\int_{0}^{(1/2)\ln 3} \frac{e^x\,dx}{1+e^{2x}}$

35. $\int_{3/4}^{4/5} \frac{dx}{x^2\sqrt{1-x^2}}$

36. $\int \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

37. $\int \frac{dx}{(a^2-x^2)^{3/2}}$

38. $\int \frac{\sin\theta\,d\theta}{\sqrt{2-\cos^2\theta}}$

39. Evaluate

$$\int \frac{y\,dy}{\sqrt{16-y^2}}$$

a) without a trigonometric substitution,
b) with a trigonometric substitution.

7.5 TRIGONOMETRIC SUBSTITUTIONS

1. Let $x = 2\tan u \Rightarrow dx = 2\sec^2 u\,du$

$$\int_{-2}^{2} \frac{dx}{4+x^2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2\sec^2 u\,du}{4+4\tan^2 u} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\,du = \frac{1}{2}u\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{4}$$

3. $\displaystyle\int \frac{dx}{1+4x^2} = \frac{1}{2}\int \frac{2\,dx}{1+(2x)^2} = \frac{1}{2}\tan^{-1} 2x + C$

5. $\displaystyle\int_0^{\frac{\pi}{8}} \frac{2\,dx}{\sqrt{1-4x^2}} = \sin^{-1}(2x)\Big]_0^{\frac{\pi}{8}} = \frac{\sqrt{2}}{2}$

7. Let $5\tan u = y \Rightarrow 5\sec^2 u\,du = dy$. $\displaystyle\int \frac{dy}{\sqrt{25+y^2}} = \int \frac{5\sec^2 u\,du}{\sqrt{25+25\tan^2 u}}$

$$= \int \sec u\,du = \ln|\sec u + \tan u| = \ln\left|\frac{\sqrt{25+y^2}}{5} + \frac{y}{5}\right| + C$$

9. Let $5\tan u = 3y \Rightarrow 5\sec^2 u\,du = 3\,dy$. $\displaystyle\int \frac{dy}{\sqrt{25+9y^2}} = \int \frac{\frac{5}{3}\sec^2 u\,du}{\sqrt{25+25\tan^2 u}}$

$$= \frac{1}{3}\int \sec u\,du = \frac{1}{3}\ln|\sec u + \tan u| = \frac{1}{3}\ln\left|\frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5}\right| + C$$

(Note: the denominator may be absorbed into the constant)

11. Let $3z = \sec u \Rightarrow 3\,dz = \sec u\tan u\,du$. Then $\displaystyle\int \frac{3\,dz}{\sqrt{9z^2-1}} = \int \frac{\sec u\tan u\,du}{\sqrt{\sec^2 u-1}}$

$$= \int \sec u\,du = \ln|\sec u + \tan u| = \ln\left|3z + \sqrt{9z^2-1}\right| + C$$

13. Let $x = 4\sec u \Rightarrow dx = 4\sec u\tan u\,du$. Then $\displaystyle\int_{\frac{8}{\sqrt{3}}}^{8} \frac{dx}{x\sqrt{x^2-16}}$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sec u\tan u\,du}{4\sec u\sqrt{16\sec^2 u-16}} = \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} du = \frac{1}{4}u\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{24}$$

15. Let $x = \frac{1}{2}\sec u \Rightarrow dx = \frac{1}{2}\sec u \tan u\, du$. Then $\int_{\frac{\sqrt{2}}{2}}^{1} \frac{dx}{2x\sqrt{4x^2 - 1}}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{2}\sec u \tan u\, du}{\sec u\sqrt{\sec^2 u - 1}} = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} du = \frac{1}{2}u\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{24}$$

17. Let $x = 2\sec w \Rightarrow dx = 2\sec w \tan w dx$, $x = 4 \Rightarrow w = \frac{\pi}{3}$, $x = 2 \Rightarrow w = 0$

$$\int_2^4 \sqrt{x^2 - 4}\, dx = 2\int_0^{\frac{\pi}{3}} \sqrt{4\sec^2 w - 4}\ \sec w \tan w\, dw = 4\int_0^{\frac{\pi}{3}} \tan^2 w \sec w\, dw$$

$$= 4\int_0^{\frac{\pi}{3}} (\sec^2 w - 1)\sec w\, dw = 4\int_0^{\frac{\pi}{3}} (\sec^3 w - \sec w)\, dw$$

$$= 4\left(\frac{1}{2}\sec w \tan w + \frac{1}{2}\ln|\sec w + \tan w| - \ln|\sec w + \tan w|\right)\Big]_0^{\frac{\pi}{3}}$$

$$= 2\sec w \tan w - 2\ln|\sec w + \tan w|\Big]_0^{\frac{\pi}{3}} = 4\sqrt{3} - 2\ln(2 + \sqrt{3})$$

(For integration of $\int \sec^3 w\, dw$, see Sect. 7.3, Ex. 3)

19. Let $x = 3\sin u \Rightarrow dx = 3\cos u\, du$. Then $\int \frac{dx}{x^2\sqrt{9 - x^2}} =$

$$\int \frac{3\cos u\, du}{9\sin^2 u\sqrt{9 - 9\sin^2 u}} = \frac{1}{9}\int \csc^2 u\, du = -\frac{1}{9}\left(\frac{\sqrt{9 - x^2}}{x}\right) + C$$

21. Let $x = \csc u \Rightarrow dx = -\csc u \cot u\, du$. Then $\int_{\frac{5}{4}}^{\frac{5}{3}} \frac{dx}{x^2\sqrt{x^2 - 1}} =$

$$\int_{x=\frac{5}{4}}^{x=\frac{5}{3}} \frac{-\csc u \cot u\, du}{\csc^2 u\sqrt{\csc^2 u - 1}} = \int_{x=\frac{5}{4}}^{x=\frac{5}{3}} (-\sin u)\, du = \frac{\sqrt{x^2 - 1}}{x}\Big]_{\frac{5}{4}}^{\frac{5}{3}} = \frac{1}{5}$$

23. Let $x = 5\sin u \Rightarrow dx = 5\cos u\, du$, $x = 0 \Rightarrow u = 0$, $x = 5 \Rightarrow u = \frac{\pi}{2}$.

$$\int_0^5 \sqrt{25 - x^2}\ dx = \int_0^{\frac{\pi}{2}} \sqrt{25 - 25\sin^2 u} \cdot 5\cos u\, du = 25\int_0^{\frac{\pi}{2}} \cos^2 u\, du$$

$$= 25\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right) du = 25\left(\frac{1}{2}u + \frac{1}{4}\sin 2u\right)\Big]_0^{\frac{\pi}{2}} = \frac{25\pi}{4}$$

25. Let $x-1=2\sin u \Rightarrow dx=2\cos u\,du$, $x=2\Rightarrow u=\frac{\pi}{6}$, $x=1\Rightarrow u=0$.

$$\int_1^2 \frac{dx}{\sqrt{4-(x-1)^2}}=\int_0^{\frac{\pi}{6}}\frac{2\cos u\,du}{\sqrt{4-4\sin^2 u}}=\int_0^{\frac{\pi}{6}}du=u\Big]_0^{\frac{\pi}{6}}=\frac{\pi}{6}$$

27. Let $x=2\sin u \Rightarrow dx=2\cos u\,du$, $x=1\Rightarrow u=\frac{\pi}{6}$, $x=0\Rightarrow u=0$.

$$\int_0^1 \frac{12\,dx}{\sqrt{4-x^2}}=\int_0^{\frac{\pi}{6}}\frac{24\cos u\,du}{\sqrt{4-4\sin^2 u}}=12\int_0^{\frac{\pi}{6}}du=12\,u\Big]_0^{\frac{\pi}{6}}=2\pi$$

29. Let $x=\tan u \Rightarrow dx=\sec^2 u\,du$, $x=0\Rightarrow u=0$, $x=1\Rightarrow u=\frac{\pi}{4}$.

$$\int_0^1 \frac{x^3\,dx}{\sqrt{x^2+1}}=\int_0^{\frac{\pi}{4}}\frac{\tan^3 u\sec^2 u\,du}{\sqrt{\tan^2 u+1}}=\int_0^{\frac{\pi}{4}}\tan^3 u\sec u\,du=$$

$$\frac{1}{3}\sec^3 u-\sec u\Big]_0^{\frac{\pi}{4}}=\frac{2-\sqrt{3}}{3}$$

31. Let $x=\frac{1}{2}\sec u \Rightarrow dx=\frac{1}{2}\sec u\tan u\,du$. Then

$$\int\frac{dx}{x\sqrt{x^2-\frac{1}{4}}}=\int\frac{\frac{1}{2}\sec u\tan u\,du}{\frac{1}{2}\sec u\sqrt{\frac{1}{4}\sec^2 u-\frac{1}{4}}}=\int 2\,du=2\sec^{-1}2x+C$$

33. Let $x=\sin u \Rightarrow dx=\cos u\,du$. Then $\displaystyle\int\frac{\sqrt{1-x^2}}{x^2}dx=\int\frac{\sqrt{1-\sin^2 u}}{\sin^2 u}\cos u\,du$

$$=\int\cot^2 u\,du=-\cot u-u+C=-\frac{\sqrt{1-x^2}}{x}-\sin^{-1}x+C$$

35. $$\int_{\frac{3}{4}}^{\frac{4}{5}}\frac{dx}{x^2\sqrt{1-x^2}}=\int_{x=\frac{3}{4}}^{x=\frac{4}{5}}\frac{\cos u\,du}{\sin^2 u\sqrt{1-\sin^2 u}}=\int_{x=\frac{3}{4}}^{x=\frac{4}{5}}\csc^2 u\,du$$

$$=-\cot u\Big]_{x=\frac{3}{4}}^{x=\frac{4}{5}}=-\frac{\sqrt{1-x^2}}{x}\Big]_{x=\frac{3}{4}}^{x=\frac{4}{5}}=-\left(\frac{3}{4}-\frac{\sqrt{7}}{3}\right)$$

37. $$\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}=\int\frac{a\cos u\,du}{\left(\sqrt{a^2-a^2\sin^2 u}\right)^3}=\frac{1}{a^2}\int\sec^2 u\,du=\frac{1}{a^2}\left(\frac{x}{\sqrt{a^2-x^2}}\right)+C$$