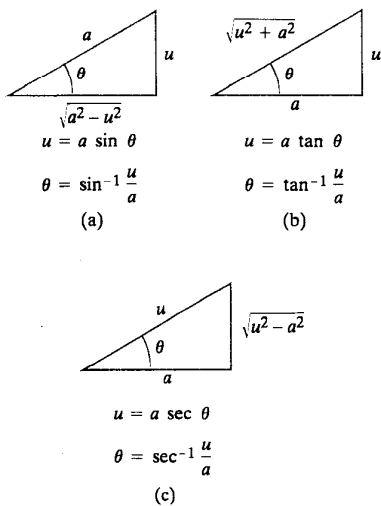


PROBLEMS



Reference triangles.

In Problems 1–15, evaluate the integrals.

- $\int_{-2}^2 \frac{dx}{4 + x^2}$
- $\int_0^2 \frac{dx}{8 + 2x^2}$
- $\int \frac{dx}{1 + 4x^2}$
- $\int_0^{3/2} \frac{dx}{\sqrt{9 - x^2}}$
- $\int_0^{1/2\sqrt{2}} \frac{2 dx}{\sqrt{1 - 4x^2}}$
- $\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9 - 4x^2}}$
- $\int \frac{dy}{\sqrt{25 + y^2}}$
- $\int \frac{3 dy}{\sqrt{1 + 9y^2}}$
- $\int \frac{dy}{\sqrt{25 + 9y^2}}$
- $\int \frac{dz}{\sqrt{z^2 - 4}}$
- $\int \frac{3 dz}{\sqrt{9z^2 - 1}}$
- $\int \frac{dz}{\sqrt{25z^2 - 9}}$
- $\int_{8/\sqrt{3}}^8 \frac{dx}{x\sqrt{x^2 - 16}}$
- $\int_2^{\sqrt{6}} \frac{dx}{x\sqrt{x^2 - 3}}$
- $\int_{\sqrt{2}/2}^1 \frac{dx}{2x\sqrt{4x^2 - 1}}$

In Problems 16–22, evaluate the integrals with the substitutions given.

- $\int \frac{dy}{y^2\sqrt{y^2 - 16}}$, $y = 4 \sec u$
- $\int_2^4 \sqrt{x^2 - 4} dx$, $x = 2 \sec u$
- $\int_0^{4/5} \frac{x^3 dx}{\sqrt{1 - x^2}}$, $x = \cos u$
- $\int \frac{dx}{x^2\sqrt{9 - x^2}}$, $x = 3 \sin u$
- $\int_0^{1/2} \frac{dx}{\sqrt{1 + x^2}}$, $x = \tan u$
- $\int_{5/4}^{5/3} \frac{dx}{x^2\sqrt{x^2 - 1}}$, $x = \csc u$
- $\int_{-1/2}^1 \frac{\sqrt{1 - x}}{x} dx$, $x = \cos^2 u$

Evaluate the integrals in Problems 23–38.

- $\int_0^5 \sqrt{25 - x^2} dx$
- $\int \frac{dx}{\sqrt{1 - 4x^2}}$
- $\int_1^2 \frac{dx}{\sqrt{4 - (x - 1)^2}}$
- $\int_0^2 \frac{dx}{\sqrt{4 + x^2}}$
- $\int_0^1 \frac{12 dx}{\sqrt{4 - x^2}}$
- $\int_0^{3/2} \frac{x dx}{\sqrt{4 + x^2}}$
- $\int_0^1 \frac{x^3 dx}{\sqrt{x^2 + 1}}$
- $\int \frac{x + 1}{\sqrt{4 - x^2}} dx$
- $\int \frac{dx}{x\sqrt{x^2 - 1/4}}$
- $\int \frac{dx}{\sqrt{2 - 5x^2}}$
- $\int \frac{\sqrt{1 - x^2}}{x^2} dx$
- $\int_0^{(1/2)\ln 3} \frac{e^x dx}{1 + e^{2x}}$
- $\int_{3/4}^{4/5} \frac{dx}{x^2\sqrt{1 - x^2}}$
- $\int \frac{4x^2 dx}{(1 - x^2)^{3/2}}$
- $\int \frac{dx}{(a^2 - x^2)^{3/2}}$
- $\int \frac{\sin \theta d\theta}{\sqrt{2 - \cos^2 \theta}}$

39. Evaluate

$$\int \frac{y dy}{\sqrt{16 - y^2}}$$

- without a trigonometric substitution,
- with a trigonometric substitution.

7.5 TRIGONOMETRIC SUBSTITUTIONS

1. Let $x = 2 \tan u \Rightarrow dx = 2 \sec^2 u \, du$

$$\int_{-2}^2 \frac{dx}{4+x^2} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2 \sec^2 u \, du}{4+4 \tan^2 u} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \, du = \frac{1}{2} u \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{4}$$

3. $\int \frac{dx}{1+4x^2} = \frac{1}{2} \int \frac{2 \, dx}{1+(2x)^2} = \frac{1}{2} \tan^{-1} 2x + C$

5. $\int_0^{\frac{\pi}{8}} \frac{2 \, dx}{\sqrt{1-4x^2}} = \sin^{-1}(2x) \Big|_0^{\frac{\pi}{8}} = \frac{\sqrt{2}}{2}$

7. Let $5 \tan u = y \Rightarrow 5 \sec^2 u \, du = dy$. $\int \frac{dy}{\sqrt{25+y^2}} = \int \frac{5 \sec^2 u \, du}{\sqrt{25+25 \tan^2 u}}$
 $= \int \sec u \, du = \ln |\sec u + \tan u| = \ln \left| \frac{\sqrt{25+y^2}}{5} + \frac{y}{5} \right| + C$

9. Let $5 \tan u = 3y \Rightarrow 5 \sec^2 u \, du = 3 \, dy$. $\int \frac{dy}{\sqrt{25+9y^2}} = \int \frac{\frac{5}{3} \sec^2 u \, du}{\sqrt{25+25 \tan^2 u}}$
 $= \frac{1}{3} \int \sec u \, du = \frac{1}{3} \ln |\sec u + \tan u| = \frac{1}{3} \ln \left| \frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5} \right| + C$

(Note: the denominator may be absorbed into the constant)

11. Let $3z = \sec u \Rightarrow 3 \, dz = \sec u \tan u \, du$. Then $\int \frac{3 \, dz}{\sqrt{9z^2-1}} = \int \frac{\sec u \tan u \, du}{\sqrt{\sec^2 u - 1}}$
 $= \int \sec u \, du = \ln |\sec u + \tan u| = \ln |3z + \sqrt{9z^2-1}| + C$

13. Let $x = 4 \sec u \Rightarrow dx = 4 \sec u \tan u \, du$. Then $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sqrt{3} x \sqrt{x^2-16}}$
 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4 \sec u \tan u \, du}{4 \sec u \sqrt{16 \sec^2 u - 16}} = \frac{1}{4} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} du = \frac{1}{4} u \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{24}$

15. Let $x = \frac{1}{2} \sec u \Rightarrow dx = \frac{1}{2} \sec u \tan u \, du$. Then $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{2x\sqrt{4x^2-1}}$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{2} \sec u \tan u \, du}{\sec u \sqrt{\sec^2 u - 1}} = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} du = \frac{1}{2} u \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{24}$$

17. Let $x = 2 \sec w \Rightarrow dx = 2 \sec w \tan w \, dw$, $x = 4 \Rightarrow w = \frac{\pi}{3}$, $x = 2 \Rightarrow w = 0$

$$\begin{aligned} \int_2^4 \sqrt{x^2-4} \, dx &= 2 \int_0^{\frac{\pi}{3}} \sqrt{4\sec^2 w - 4} \sec w \tan w \, dw = 4 \int_0^{\frac{\pi}{3}} \tan^2 w \sec w \, dw \\ &= 4 \int_0^{\frac{\pi}{3}} (\sec^2 w - 1) \sec w \, dw = 4 \int_0^{\frac{\pi}{3}} (\sec^3 w - \sec w) \, dw \\ &= 4 \left(\frac{1}{2} \sec w \tan w + \frac{1}{2} \ln |\sec w + \tan w| - \ln |\sec w + \tan w| \right) \Big|_0^{\frac{\pi}{3}} \\ &= 2 \sec w \tan w - 2 \ln |\sec w + \tan w| \Big|_0^{\frac{\pi}{3}} = 4\sqrt{3} - 2 \ln(2 + \sqrt{3}) \\ &\quad \left(\text{For integration of } \int \sec^3 w \, dw, \text{ see Sect. 7.3, Ex. 3} \right) \end{aligned}$$

19. Let $x = 3 \sin u \Rightarrow dx = 3 \cos u \, du$. Then $\int \frac{dx}{x^2 \sqrt{9-x^2}} =$

$$\int \frac{3 \cos u \, du}{9 \sin^2 u \sqrt{9-9\sin^2 u}} = \frac{1}{9} \int \csc^2 u \, du = -\frac{1}{9} \left(\frac{\sqrt{9-x^2}}{x} \right) + C$$

21. Let $x = \csc u \Rightarrow dx = -\csc u \cot u \, du$. Then $\int_{\frac{5}{4}}^{\frac{5}{3}} \frac{dx}{x^2 \sqrt{x^2-1}} =$

$$\int_{x=\frac{5}{4}}^{x=\frac{5}{3}} \frac{-\csc u \cot u \, du}{\csc^2 u \sqrt{\csc^2 u - 1}} = \int_{x=\frac{5}{4}}^{x=\frac{5}{3}} (-\sin u) \, du = \frac{\sqrt{x^2-1}}{x} \Big|_{\frac{5}{4}}^{\frac{5}{3}} = \frac{1}{5}$$

23. Let $x = 5 \sin u \Rightarrow dx = 5 \cos u \, du$, $x = 0 \Rightarrow u = 0$, $x = 5 \Rightarrow u = \frac{\pi}{2}$.

$$\begin{aligned} \int_0^5 \sqrt{25-x^2} \, dx &= \int_0^{\frac{\pi}{2}} \sqrt{25-25\sin^2 u} \cdot 5 \cos u \, du = 25 \int_0^{\frac{\pi}{2}} \cos^2 u \, du \\ &= 25 \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2} \cos 2u \right) du = 25 \left(\frac{1}{2} u + \frac{1}{4} \sin 2u \right) \Big|_0^{\frac{\pi}{2}} = \frac{25\pi}{4} \end{aligned}$$

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15. Let $x - 1 = 2 \sin u \Rightarrow dx = 2 \cos u \, du$, $x = 2 \Rightarrow u = \frac{\pi}{6}$, $x = 1 \Rightarrow u = 0$.

$$\int_1^2 \frac{dx}{\sqrt{4 - (x-1)^2}} = \int_0^{\frac{\pi}{6}} \frac{2 \cos u \, du}{\sqrt{4 - 4 \sin^2 u}} = \int_0^{\frac{\pi}{6}} du = u \Big|_0^{\frac{\pi}{6}} = \frac{\pi}{6}$$

17. Let $x = 2 \sin u \Rightarrow dx = 2 \cos u \, du$, $x = 1 \Rightarrow u = \frac{\pi}{6}$, $x = 0 \Rightarrow u = 0$.

$$\int_0^1 \frac{12 \, dx}{\sqrt{4 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{24 \cos u \, du}{\sqrt{4 - 4 \sin^2 u}} = 12 \int_0^{\frac{\pi}{6}} du = 12 u \Big|_0^{\frac{\pi}{6}} = 2\pi$$

29. Let $x = \tan u \Rightarrow dx = \sec^2 u \, du$, $x = 0 \Rightarrow u = 0$, $x = 1 \Rightarrow u = \frac{\pi}{4}$.

$$\int_0^1 \frac{x^3 \, dx}{\sqrt{x^2 + 1}} = \int_0^{\frac{\pi}{4}} \frac{\tan^3 u \sec^2 u \, du}{\sqrt{\tan^2 u + 1}} = \int_0^{\frac{\pi}{4}} \tan^3 u \sec u \, du =$$

$$\frac{1}{3} \sec^3 u - \sec u \Big|_0^{\frac{\pi}{4}} = \frac{2 - \sqrt{3}}{3}$$

31. Let $x = \frac{1}{2} \sec u \Rightarrow dx = \frac{1}{2} \sec u \tan u \, du$. Then

$$\int \frac{dx}{x \sqrt{x^2 - \frac{1}{4}}} = \int \frac{\frac{1}{2} \sec u \tan u \, du}{\frac{1}{2} \sec u \sqrt{\frac{1}{4} \sec^2 u - \frac{1}{4}}} = \int 2 \, du = 2 \sec^{-1} 2x + C$$

33. Let $x = \sin u \Rightarrow dx = \cos u \, du$. Then $\int \frac{\sqrt{1-x^2}}{x^2} \, dx = \int \frac{\sqrt{1-\sin^2 u}}{\sin^2 u} \cos u \, du$

$$= \int \cot^2 u \, du = -\cot u - u + C = -\frac{\sqrt{1-x^2}}{x} - \sin^{-1} x + C$$

35. $\int_{\frac{3}{4}}^{\frac{4}{5}} \frac{dx}{x^2 \sqrt{1-x^2}} = \int_{x=\frac{3}{4}}^{x=\frac{4}{5}} \frac{\cos u \, du}{\sin^2 u \sqrt{1-\sin^2 u}} = \int_{x=\frac{3}{4}}^{x=\frac{4}{5}} \csc^2 u \, du$

$$= -\cot u \Big|_{x=\frac{3}{4}}^{x=\frac{4}{5}} = -\frac{\sqrt{1-x^2}}{x} \Big|_{x=\frac{3}{4}}^{x=\frac{4}{5}} = -\left(\frac{3}{4} - \frac{\sqrt{7}}{3}\right)$$

37. $\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \int \frac{a \cos u \, du}{\left(\sqrt{a^2 - a^2 \sin^2 u}\right)^3} = \frac{1}{a^2} \int \sec^2 u \, du = \frac{1}{a^2} \left(\frac{x}{\sqrt{a^2 - x^2}}\right) + C$